Published by Institute of Physics Publishing for SISSA

Received: *September 6, 2007*
Revised: *November 29, 2007*
Accepted: *December 12, 2007*
Published: *December 19, 2007*

On the fusion matrix of the $N = 1$ Neveu-Schwarz blocks

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Abstract: We propose an exact form of the fusion matrix of the Neveu-Schwarz blocks that appear in a correlation function of four super-primary fields. Orthogonality relation satisfied by this matrix is equivalent to the bootstrap equation for the four-point super-primary correlator in the $N = 1$ supersymmetric Liouville field theory.

Keywords: Field Theories in Lower Dimensions, Conformal and W Symmetry, Quantum Groups.

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1. Introduction

In the BPZ formulation of the conformal field theory [1] the basic dynamical principle is an associativity of the operator product algebra. Its direct consequence is the bootstrap equation for the four-point correlation function [1]. Once a three point coupling constants of a given CFT are known, the bootstrap equation can be viewed as the basic consistency condition of the theory.

The simplest CFT with a continuous spectrum, which cannot be obtained from a free field theory in a simple way, is the Liouville theory. Its three point coupling constants have been found independently by Dorn and Otto [2] and by Zamolodchikov and Zamolodchikov [3]. The authors of [3] also performed a numerical check of the bootstrap equation in the Liouville theory using a recursive representation of conformal blocks developed in [4 − 6].

An analytic proof of this equation which combined a Moore-Seiberg formalism of CFT [7] with a representation theory of quantum groups has been presented in [8, 9]. Using the results on fusion of degenerate representation of the Virasoro algebra with generic ones the authors of [8, 9] derived from the consistency conditions of the Moore-Seiberg type a

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set of functional equations for the fusion coefficient of the conformal blocks. These equations were then shown to be satisfied by the Racah-Wiegner coefficients for an appropriate continuous series of representations of $U_q(\mathrm{sl}(2,\mathbb{R}))$.

Conformal field theory with $N=1$ supersymmetry [10−12] can be viewed as the simplest generalization of the "ordinary" CFT. Also here the three point coupling constants of the basic interacting model — supersymmetric extension of the Liouville theory — are known [13] and some numerical checks of the bootstrap equation in the Neveu-Schwarz sector of the theory (which employed a recursive representation of $N=1$ NS blocks developed in [14, 15]) have been performed [16, 17]. An analytical proof of the consistency of the $N=1$ supersymmetric Liouville theory is however still missing.

A goal of this paper is to make a step towards such a proof. We put forward a conjecture on an exact form of the fusion matrix for the two NS blocks that appear in a correlation function of four super-primary fields and check several of its properties. As one of the main results of the paper one may regard identities satisfied by the "supersymmetric" extensions of the basic hypergeometric functions [18], derived in section 5. These identities provide a technical tool which should allow to complete the proof of the consistency of $N=1$ supersymmetric Liouville theory (perhaps along the lines mentioned in the last section).

The paper is organized as follows. In section 2 we rewrite the bootstrap equation for the four-point correlation functions of super-primary fields in $N=1$ Liouville theory in the form of an orthogonality relation for the fusion matrix of $N=1$ Neveu-Schwarz blocks. In section 3 we construct — guided by an analogy with the form of the fusion matrix for "ordinary" Liouville blocks[1] — a conjectured fusion matrix for the basic NS blocks. In section 4 we prove some of its most important properties: orthogonality, symmetry properties and calculate its form in the case which corresponds to a degenerate representation of the NS algebra. Section 5 contains proofs of several identities satisfied by a "supersymmetric extensions" of the basic hypergeometric functions: integral analogs of the Ramanujan summation formula, Heine and Euler-Heine transformations and analog of Saalschütz summation formulae. We end up with some discussion of possible applications of the obtained results.

2. Bootstrap in the $N=1$ supersymmetric Liouville theory

Local properties of the $N=1$ supersymmetric Liouville field theory (see [19] for a comprehensive review) are described by the action

$$\mathcal{S}_{\text{SLFT}} = \int d^2z \left(\frac{1}{2\pi}\,|\partial\phi|^2 + \frac{1}{2\pi}\left(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}\right) + 2i\mu b^2\bar{\psi}\psi \mathrm{e}^{b\phi} + 2\pi b^2\mu^2\mathrm{e}^{2b\phi}\right), \tag{2.1}$$

where μ denotes a two-dimensional cosmological constant and b is a Liouville coupling constant.

The superconformal symmetry of the theory is generated by a pair of holomorphic currents $T(z)$, $S(z)$ and their anti-holomorphic counterparts $\overline{T}(\bar{z})$, $\overline{S}(\bar{z})$, where T and $\overline{T}$

[1]This analogy extends in fact to other objects in supersymmetric and "ordinary" Liouville theory, including reflection amplitudes, boundary two point functions and "bulk" one point functions in the disc and in the Lobachevsky plane geometries, see for instance [19].

are components of the energy-momentum tensor while S and $\overline{S}$ have dimensions $(3/2, 0)$ and $(0, 3/2)$, respectively. The modes of T and S satisfy the algebra:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12} m\left(m^2-1\right)\delta_{m+n}, \\ [L_m, S_k] &= \frac{m-2k}{2} S_{m+k}, \\ \{S_k, S_l\} &= 2L_{k+l} + \frac{c}{3}\left(k^2 - \frac{1}{4}\right)\delta_{k+l}, \end{aligned} \tag{2.2}$$

with $m, n \in \mathbb{Z}$, while $k, l \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwarz sector and $k, l \in \mathbb{Z}$ in the Ramond sector.

In the present paper we shall discuss only the Neveu-Schwarz, i.e. local with respect to $S(z)$, fields. In the space of NS fields there exit "super-primary" fields, realized in the supersymmetric Liouville field theory as an appropriately normal ordered exponents $V_\alpha(z, \bar{z}) = \mathrm{e}^{\alpha\varphi(z,\bar{z})}$. They satisfy:

$$\begin{aligned} [L_n, V_\alpha(0,0)] = [S_k, V_\alpha(0,0)] &= 0, & n, k &> 0, \\ [L_0, V_\alpha(0,0)] = \Delta_\alpha V_\alpha(0,0), & & \Delta_\alpha &= \frac{1}{2}\alpha(Q-\alpha), \end{aligned} \tag{2.3}$$

(and similarly for the "right" generators $\bar{L}_n$ and $\bar{S}_k$) where $Q = b + b^{-1}$ is the background charge related to the central charge of algebra as

$$c = \frac{3}{2} + 3Q^2.$$

Each super-primary field is the "lowest" component of the superfield

$$\Phi_\alpha(z, \theta; \bar{z}, \bar{\theta}) \ = \ V_\alpha(z, \bar{z}) + \theta\,\Lambda_\alpha(z, \bar{z}) + \bar{\theta}\,\bar{\Lambda}_\alpha(z, \bar{z}) - \theta\bar{\theta}\,\widetilde{V}_\alpha(z, \bar{z}), \tag{2.4}$$

where

$$\Lambda_\alpha \ = \ \left[S_{-1/2}, V_\alpha\right], \quad \bar{\Lambda}_\alpha \ = \ \left[\bar{S}_{-1/2}, V_\alpha\right], \quad \widetilde{V}_\alpha \ = \ \left\{S_{-1/2}, \left[\bar{S}_{-1/2}, V_\alpha\right]\right\},$$

and $\theta, \bar{\theta}$ are Grassman numbers. Global superconformal transformations (generated by $L_0, S_{\pm\frac{1}{2}}, L_{\pm 1}$ and their right counterparts) allow to express three-point function of primary superfields in the form:

$$\begin{aligned} &\left\langle \Phi_{\alpha_3}(z_3, \theta_3; \bar{z}_3, \bar{\theta}_3)\Phi_{\alpha_2}(z_2, \theta_2; \bar{z}_2, \bar{\theta}_2)\Phi_{\alpha_1}(z_1, \theta_1; \bar{z}_1, \bar{\theta}_1)\right\rangle \\ &\quad = Z_{32}^{\gamma_1}\bar{Z}_{32}^{\bar{\gamma}_1}\, Z_{31}^{\gamma_2}\bar{Z}_{31}^{\bar{\gamma}_2}\, Z_{21}^{\gamma_3}\bar{Z}_{21}^{\bar{\gamma}_3} \left\langle \Phi_{\alpha_3}(\infty, 0; \infty, 0)\Phi_{\alpha_2}(1, \Theta; 1, \bar{\Theta})\Phi_{\alpha_1}(0, 0; 0, 0)\right\rangle, \end{aligned} \tag{2.5}$$

where $\gamma_i = 2\Delta_i - (\Delta_1 + \Delta_2 + \Delta_3)$, $Z_{ij} = z_i - z_j - \theta_i\theta_j \equiv z_{ij} - \theta_i\theta_j$,

$$\Theta \ = \ \frac{1}{\sqrt{z_{12}z_{13}z_{23}}}\left(\theta_1 z_{23} + \theta_2 z_{31} + \theta_3 z_{12} - \frac{1}{2}\theta_1\theta_2\theta_3\right),$$

and

$$\Phi_{\alpha_3}(\infty, 0; \infty, 0) \equiv \lim_{R\to\infty} R^{2\Delta_3 + 2\bar{\Delta}_3}\Phi_{\alpha_3}(R, 0; R, 0).$$

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The three point function is thus determined by the superconformal symmetry up to two independent constants,

$$C(\alpha_3,\alpha_2,\alpha_1) = \Big\langle V_{\alpha_3}(\infty,\infty)V_{\alpha_2}(1,1)V_{\alpha_1}(0,0)\Big\rangle,$$
$$\widetilde{C}(\alpha_3,\alpha_2,\alpha_1) = \Big\langle V_{\alpha_3}(\infty,\infty)\widetilde{V}_{\alpha_2}(1,1)V_{\alpha_1}(0,0)\Big\rangle.$$

Their form in the $N=1$ supersymmetric Liouville field theory,[2]

$$C(\alpha_3,\alpha_2,\alpha_1) = C_0(\alpha)\,\frac{\Upsilon_{\rm NS}(2\alpha_3)\Upsilon_{\rm NS}(2\alpha_2)\Upsilon_{\rm NS}(2\alpha_1)}{\Upsilon_{\rm NS}(\alpha-Q)\Upsilon_{\rm NS}(\alpha_{1+2-3})\Upsilon_{\rm NS}(\alpha_{2+3-1})\Upsilon_{\rm NS}(\alpha_{3+1-2})},$$
$$\widetilde{C}(\alpha_3,\alpha_2,\alpha_1) = i\,C_0(\alpha)\,\frac{\Upsilon_{\rm NS}(2\alpha_3)\Upsilon_{\rm NS}(2\alpha_2)\Upsilon_{\rm NS}(2\alpha_1)}{\Upsilon_{\rm R}(\alpha-Q)\Upsilon_{\rm R}(\alpha_{1+2-3})\Upsilon_{\rm R}(\alpha_{2+3-1})\Upsilon_{\rm R}(\alpha_{3+1-2})},$$

with

$$C_0(\alpha) = \left(\pi\mu\gamma\left(\frac{bQ}{2}\right)b^{1-b^2}\right)^{\frac{Q-\alpha}{b}}\Upsilon'_{\rm NS}(0),$$

was first derived in [13]. Here $\alpha\equiv\alpha_1+\alpha_2+\alpha_3$, $\alpha_{1+2-3}\equiv\alpha_1+\alpha_2-\alpha_3$, etc. and the special functions involved ($\Upsilon_{\rm NS,R}(x),\Gamma_{\rm NS,R}(x)$ below etc.) are defined in appendix A.

It is convenient to combine C and $\tilde{C}$ in the matrix notation

$$\mathsf{C}(\alpha_3,\alpha_2,\alpha_1) = \begin{pmatrix} C(\alpha_3,\alpha_2,\alpha_1) & 0 \\ 0 & \widetilde{C}(\alpha_s,\alpha_2,\alpha_1)\end{pmatrix}.$$

Let us define:

$$\vec{\mathcal{F}}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z) = \left(\mathcal{F}^{\rm e}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z),\mathcal{F}^{\rm o}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)\right),$$

where $\mathcal{F}^{\rm e}_{\alpha_s}$ and $\mathcal{F}^{\rm o}_{\alpha_s}$ denote an even and an odd $N=1$ Neveu-Schwarz block [16, 14].

Four-point correlation function of super-primary NS fields in the $N=1$ supersymmetric Liouville field theory,

$$G_4(z,\bar{z}) = \Big\langle V_{\alpha_4}(\infty,\infty)V_{\alpha_3}(1,1)V_{\alpha_2}(z,\bar{z})V_{\alpha_1}(0,0)\Big\rangle,$$

can be written either in the "$s-$channel" representation:

$$\begin{aligned}G_4(z,\bar{z}) &= \int\limits_{\frac{Q}{2}+i\mathbb{R}_+}\frac{d\alpha_s}{i}\Big[C(\alpha_4,\alpha_3,\alpha_s)C(\bar{\alpha}_s,\alpha_2,\alpha_1)\left|\mathcal{F}^{\rm e}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)\right|^2\\ &\qquad\qquad -\widetilde{C}(\alpha_4,\alpha_3,\alpha_s)\widetilde{C}(\bar{\alpha}_s,\alpha_2,\alpha_1)\left|\mathcal{F}^{\rm o}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)\right|^2\Big]\\ &= \int\limits_{\frac{Q}{2}+i\mathbb{R}_+}\frac{d\alpha_s}{i}\,\vec{\mathcal{F}}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)\,\mathsf{C}(\alpha_4,\alpha_3,\alpha_s)\cdot\sigma_3\cdot\mathsf{C}(\bar{\alpha}_s,\alpha_2,\alpha_1)\left(\vec{\mathcal{F}}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)\right)^{\dagger},\end{aligned}$$

or in the "$t-$channel" representation:

$$G_4(z,\bar{z}) = \int\limits_{\frac{Q}{2}+i\mathbb{R}_+}\frac{d\alpha_t}{i}\,\vec{\mathcal{F}}_{\alpha_t}\left[\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_4 & \alpha_3\end{smallmatrix}\right](1-z)\,\mathsf{C}(\alpha_4,\alpha_t,\alpha_1)\cdot\sigma_3\cdot\mathsf{C}(\bar{\alpha}_t,\alpha_3,\alpha_2)\left(\vec{\mathcal{F}}_{\alpha_t}\left[\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_4 & \alpha_3\end{smallmatrix}\right](1-z)\right)^{\dagger}.$$

[2]The constant $\tilde{C}$ differs from a corresponding constant in [16] by a factor of $\frac{1}{2}$, and consequently an odd NS block differs by a factor of 2 from the conventions of [16].

Here and in what follows we use a convenient notation $\bar{\alpha} = Q - \alpha$ (notice that for $\alpha \in \frac{Q}{2} + i\mathbb{R}$ it is indeed the complex conjugate of α) and $\sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Coincidence of the $s-$ and $t-$channel representations of the four-point correlation function constitute the bootstrap equation for the super-primary fields in the supersymmetric Liouville field theory (SLFT).

Defining the SLFT fusion matrix $\mathsf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ through the equation:

$$\vec{\mathcal{F}}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z) \;=\; \int\limits_{\frac{Q}{2}+i\mathbb{R}_+} \frac{d\alpha_t}{i}\, \vec{\mathcal{F}}_{\alpha_t}\left[\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_4 & \alpha_3\end{smallmatrix}\right](1-z)\, \mathsf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right], \tag{2.6}$$

we can rewrite the bootstrap equation in the form of an orthogonality relation:

$$\begin{aligned}\int\limits_{\frac{Q}{2}+i\mathbb{R}_+} \frac{d\alpha_s}{i}\, \mathsf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] \cdot \mathsf{C}(\alpha_4,\alpha_3,\alpha_s)\cdot\sigma_3\cdot \mathsf{C}(\bar{\alpha}_s,\alpha_2,\alpha_1)\cdot \left(\mathsf{F}_{\alpha_s\alpha_t'}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\right)^{\dagger} \\ = \mathsf{C}(\alpha_4,\alpha_t,\alpha_1)\cdot\sigma_3\cdot \mathsf{C}(\bar{\alpha}_t,\alpha_3,\alpha_2)\; i\delta(\alpha_t-\alpha_t').\end{aligned} \tag{2.7}$$

Let

$$\begin{aligned}
\mathcal{N}_{\rm NS}(\alpha_3,\alpha_2,\alpha_1) &= \frac{\Gamma_{\rm NS}(2Q-2\alpha_3)\Gamma_{\rm NS}(2\alpha_2)\Gamma_{\rm NS}(2\alpha_1)}{\Gamma_{\rm NS}(2Q-\alpha_{3+2+1})\Gamma_{\rm NS}(\alpha_{3+2-1})\Gamma_{\rm NS}(\alpha_{2+1-3})\Gamma_{\rm NS}(\alpha_{1+3-2})},\\
\mathcal{N}_{\rm R}(\alpha_3,\alpha_2,\alpha_1) &= \frac{\Gamma_{\rm NS}(2Q-2\alpha_3)\Gamma_{\rm NS}(2\alpha_2)\Gamma_{\rm NS}(2\alpha_1)}{\Gamma_{\rm R}(2Q-\alpha_{3+2+1})\Gamma_{\rm R}(\alpha_{3+2-1})\Gamma_{\rm R}(\alpha_{2+1-3})\Gamma_{\rm R}(\alpha_{1+3-2})},\\
\mathsf{N}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] &= \begin{pmatrix}\mathcal{N}_{\rm NS}(\alpha_4,\alpha_3,\alpha_s)\mathcal{N}_{\rm NS}(\alpha_s,\alpha_2,\alpha_1) & 0\\ 0 & \mathcal{N}_{\rm R}(\alpha_4,\alpha_3,\alpha_s)\mathcal{N}_{\rm R}(\alpha_s,\alpha_2,\alpha_1)\end{pmatrix},
\end{aligned} \tag{2.8}$$

and

$$\mathsf{G}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \mathsf{N}_{\alpha_t}\left[\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_4 & \alpha_3\end{smallmatrix}\right]\cdot \mathsf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\cdot \left(\mathsf{N}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\right)^{-1}. \tag{2.9}$$

These definitions are motivated by an identity:

$$\begin{aligned}&\mathsf{N}_s\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] \mathsf{C}(\alpha_4,\alpha_3,\alpha_s)\cdot\sigma_3\cdot\mathsf{C}(\bar{\alpha}_s,\alpha_2,\alpha_1)\left(\mathsf{N}_s\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\right)^{\dagger}\\ &\quad = \left(\pi\mu\gamma\left(\frac{bQ}{2}\right)b^{1-b^2}\right)^{\frac{Q-\alpha_{1+2+3+4}}{b}} \left(\Upsilon_{\rm NS}'(0)\right)^2 |S_{\rm NS}(2\alpha_s)|^2\,\sigma_0,\end{aligned} \tag{2.10}$$

where $\sigma_0 \equiv \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$. In view of (2.10) we can rewrite the equation (2.7) as a simple orthogonality relation for the matrix G :

$$\int\limits_{\frac{Q}{2}+i\mathbb{R}_+}\frac{d\alpha_s}{i}\; |S_{\rm NS}(2\alpha_s)|^2\; \mathsf{G}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\cdot\left(\mathsf{G}_{\alpha_s\alpha_t'}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\right)^{\dagger} = |S_{\rm NS}(2\alpha_t)|^2\,\sigma_0\; i\delta(\alpha_t-\alpha_t'). \tag{2.11}$$

The fusion matrix $\mathsf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ is expected [7] to be invariant with respect to the separate conjugations $\alpha_i \to Q-\alpha_i$ of all six of its arguments (and thus to depend only on the conformal weights $\Delta_i = \frac{1}{2}\alpha_i(Q-\alpha_i)$) and should not change under exchange of its rows and columns,

$$\mathsf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] \;=\; \mathsf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_4 & \alpha_1\\ \alpha_3 & \alpha_2\end{smallmatrix}\right] \;=\; \mathsf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_2 & \alpha_3\\ \alpha_1 & \alpha_4\end{smallmatrix}\right].$$

3. Explicit form of the fusion matrix

In analogy with [8, 9] we shall define a "supersymmetric", deformed hypergeometric function:

$$\mathbb{F}(\alpha,\beta;\gamma;z) = \begin{pmatrix} F^{(+)}_{\rm NS}(\alpha,\beta;\gamma;z) & F^{(-)}_{\rm R}(\alpha,\beta;\gamma;z) \\ F^{(-)}_{\rm NS}(\alpha,\beta;\gamma;z) & F^{(+)}_{\rm R}(\alpha,\beta;\gamma;z) \end{pmatrix}, \tag{3.1}$$

where:[3]

$$\begin{aligned} F^{(\pm)}_{\rm NS}(\alpha,\beta;\gamma;z) &= \frac{S_{\rm NS}(\gamma)}{S_{\rm NS}(\alpha)S_{\rm NS}(\beta)} \\ &\quad\times \int_{-i\infty}^{i\infty} \frac{d\tau}{i}\, {\rm e}^{i\pi\tau z} \left[\frac{S_{\rm NS}(\tau+\alpha)S_{\rm NS}(\tau+\beta)}{S_{\rm NS}(\tau+\gamma)S_{\rm NS}(\tau+Q)} \pm \frac{S_{\rm R}(\tau+\alpha)S_{\rm R}(\tau+\beta)}{S_{\rm R}(\tau+\gamma)S_{\rm R}(\tau+Q)}\right], \\ F^{(\pm)}_{\rm R}(\alpha,\beta;\gamma;z) &= \frac{S_{\rm NS}(\gamma)}{S_{\rm R}(\alpha)S_{\rm R}(\beta)} \\ &\quad\times \int_{-i\infty}^{i\infty} \frac{d\tau}{i}\, {\rm e}^{i\pi\tau z} \left[\frac{S_{\rm NS}(\tau+\alpha)S_{\rm NS}(\tau+\beta)}{S_{\rm R}(\tau+\gamma)S_{\rm R}(\tau+Q)} \pm \frac{S_{\rm R}(\tau+\alpha)S_{\rm R}(\tau+\beta)}{S_{\rm NS}(\tau+\gamma)S_{\rm NS}(\tau+Q)}\right]. \end{aligned}$$

It satisfies an important relation:

$$\mathbb{F}(\alpha,\beta;\gamma;z) = {\rm e}^{\frac{i\pi}{2}(\gamma-\alpha-\beta)z} \begin{pmatrix} \frac{S_{\rm NS}(z+\frac{Q+\gamma-\alpha-\beta}{2})}{S_{\rm NS}(z+\frac{Q-\gamma+\alpha+\beta}{2})} & 0 \\ 0 & \frac{S_{\rm R}(z+\frac{Q+\gamma-\alpha-\beta}{2})}{S_{\rm R}(z+\frac{Q-\gamma+\alpha+\beta}{2})} \end{pmatrix} \mathbb{F}(\gamma-\alpha,\gamma-\beta;\gamma;z), \tag{3.2}$$

which arises by expressing (5.7) through the functions $S_{\rm NS}$ and $S_{\rm R}$.

Let us further define:

$$\begin{aligned} \Theta_s(x|\alpha_s) &= \frac{1}{4\sqrt{2}}\,{\rm e}^{\pi x(\frac{Q}{2}+\alpha_s-\alpha_1-\alpha_2)} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbb{F}(\alpha_s+\alpha_1-\alpha_2,\alpha_s+\alpha_3-\bar\alpha_4;2\alpha_s;-ix), \\ \Theta_t(x|\alpha_t) &= \frac{1}{4\sqrt{2}}\,{\rm e}^{-\pi x(\alpha_t+\alpha_1-\bar\alpha_4-\frac{Q}{2})} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbb{F}(\alpha_t+\alpha_1-\bar\alpha_4,\alpha_t+\alpha_3-\alpha_2;2\alpha_t;ix). \end{aligned} \tag{3.3}$$

Explicitly,

$$\Theta_s(x|\alpha_s) = \frac{1}{2\sqrt{2}}\,S_{\rm NS}(2\alpha_s) \begin{pmatrix} \theta^s_{\rm NN}(x|\alpha_s) & \theta^s_{\rm NR}(x|\alpha_s) \\ \theta^s_{\rm RR}(x|\alpha_s) & -\theta^s_{\rm RN}(x|\alpha_s) \end{pmatrix} \cdot \mathbf{U}_s(\alpha_s),$$

with

$$\mathbf{U}_s(\alpha_s) = \begin{pmatrix} S_{\rm NS}(\alpha_s+\alpha_1-\alpha_2)S_{\rm NS}(\alpha_s+\alpha_3-\bar\alpha_4) & 0 \\ 0 & S_{\rm R}(\alpha_s+\alpha_1-\alpha_2)S_{\rm R}(\alpha_s+\alpha_3-\bar\alpha_4) \end{pmatrix}^{-1}$$

[3]The contour of integration is located to the right of the poles of the integrand coming from the poles of the functions in the numerator and to the left from the poles of the integrand due to zeroes of the functions in the denominator; see definitions in section 5 and appendix A.

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and

$$
\begin{aligned}
\theta^s_{\rm NN}(x|\alpha_s) &= \int\limits_{-i\infty}^{i\infty} \frac{d\tau}{i}\; \mathrm{e}^{\pi x(\frac{Q}{2}+\tau+\alpha_s-\alpha_1-\alpha_2)}\; \frac{S_{\rm NS}(\tau+\alpha_s+\alpha_1-\alpha_2)S_{\rm NS}(\tau+\alpha_s+\alpha_3-\bar{\alpha}_4)}{S_{\rm NS}(\tau+2\alpha_s)S_{\rm NS}(\tau+Q)},\\
\theta^s_{\rm NR}(x|\alpha_s) &= \int\limits_{-i\infty}^{i\infty} \frac{d\tau}{i}\; \mathrm{e}^{\pi x(\frac{Q}{2}+\tau+\alpha_s-\alpha_1-\alpha_2)}\; \frac{S_{\rm NS}(\tau+\alpha_s+\alpha_1-\alpha_2)S_{\rm NS}(\tau+\alpha_s+\alpha_3-\bar{\alpha}_4)}{S_{\rm R}(\tau+2\alpha_s)S_{\rm R}(\tau+Q)},\\
\theta^s_{\rm RN}(x|\alpha_s) &= \int\limits_{-i\infty}^{i\infty} \frac{d\tau}{i}\; \mathrm{e}^{\pi x(\frac{Q}{2}+\tau+\alpha_s-\alpha_1-\alpha_2)}\; \frac{S_{\rm R}(\tau+\alpha_s+\alpha_1-\alpha_2)S_{\rm R}(\tau+\alpha_s+\alpha_3-\bar{\alpha}_4)}{S_{\rm NS}(\tau+2\alpha_s)S_{\rm NS}(\tau+Q)},\\
\theta^s_{\rm RR}(x|\alpha_s) &= \int\limits_{-i\infty}^{i\infty} \frac{d\tau}{i}\; \mathrm{e}^{\pi x(\frac{Q}{2}+\tau+\alpha_s-\alpha_1-\alpha_2)}\; \frac{S_{\rm R}(\tau+\alpha_s+\alpha_1-\alpha_2)S_{\rm R}(\tau+\alpha_s+\alpha_3-\bar{\alpha}_4)}{S_{\rm R}(\tau+2\alpha_s)S_{\rm R}(\tau+Q)}.
\end{aligned}
\tag{3.4}
$$

Notice that $\mathbf{U}_s(\alpha_s)$ is $x-$independent and (for $\alpha_s, \alpha_i \in \frac{Q}{2}+i\mathbb{R}$) unitary,

$$
\mathbf{U}_s^\dagger(\alpha_s)\mathbf{U}_s(\alpha_s) \;=\; \mathbf{U}_s(\alpha_s)\mathbf{U}_s^\dagger(\alpha_s) = \sigma_0.
$$

Finally, define (unitary) normalization factors:

$$
\begin{aligned}
M^s_\flat\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] &= \frac{S_\flat(\alpha_s+\alpha_3-\alpha_4)S_\flat(\alpha_s-\alpha_1+\alpha_2)S_\flat(\alpha_s+\alpha_1-\alpha_2)}{S_\flat(\alpha_s-\alpha_1+\bar{\alpha}_2)}, \qquad \flat = \text{NS, R},\\
M^t_\flat\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] &= \frac{S_\flat(\alpha_t+\alpha_1-\alpha_4)S_\flat(\alpha_t-\alpha_3+\alpha_2)S_\flat(\alpha_t+\alpha_3-\alpha_2)}{S_\flat(\alpha_t-\alpha_3+\bar{\alpha}_2)},\\
\mathbf{M}^\natural_{\alpha_\natural}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] &= \begin{pmatrix} M^\natural_{\rm NS}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] & 0\\ 0 & M^\natural_{\rm R}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\end{pmatrix}, \qquad \natural = s,\, t.
\end{aligned}
\tag{3.5}
$$

Let

$$
\mathbf{G}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \frac{S_{\rm NS}(2\alpha_t)}{S_{\rm NS}(2\alpha_s)}\left(\mathbf{M}^t_{\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\right)^\dagger \left(\int\limits_{\mathbb{R}} dx\; \Theta^\dagger_t(x|\alpha_t)\Theta_s(x|\alpha_s)\right)\mathbf{M}^s_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right], \tag{3.6}
$$

and

$$
\mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \left(\mathsf{N}_{\alpha_t}\left[\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_4 & \alpha_3\end{smallmatrix}\right]\right)^{-1}\cdot\mathbf{G}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\cdot\mathsf{N}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right], \tag{3.7}
$$

where the normalization factors $\mathsf{N}_s\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ are defined in eq. (2.8).

We expect that an equality:

$$
\mathsf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right], \tag{3.8}
$$

where $\mathsf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ is the fusion matrix defined in eq. (2.6), holds. In the next section we shall present some arguments in favor of (3.8).

4. Properties of the matrix F

4.1 Orthogonality

A short calculation allows to check an identity:

$$\int_{\mathbb{R}} dx\ \Theta_s^\dagger(x|\alpha_s)\Theta_s(x|\alpha_s') = \frac{1}{4}\overline{S_{\rm NS}(2\alpha_s)}S_{\rm NS}(2\alpha_s')$$

$$\times \mathbf{U}_s^\dagger(\alpha_s)\int_{-i\infty}^{i\infty}\frac{d\tau}{i}\begin{pmatrix}\langle\tau\,|^{\rm N}_{\rm N}|\,\xi_s\rangle & \langle\tau\,|^{\rm R}_{\rm R}|\,\xi_s\rangle\\ \langle\tau\,|^{\rm R}_{\rm R}|\,\xi_s\rangle & -\langle\tau\,|^{\rm N}_{\rm N}|\,\xi_s\rangle\end{pmatrix}^\dagger\begin{pmatrix}\langle\tau\,|^{\rm N}_{\rm N}|\,\xi_s'\rangle & \langle\tau\,|^{\rm R}_{\rm R}|\,\xi_s'\rangle\\ \langle\tau\,|^{\rm R}_{\rm R}|\,\xi_s'\rangle & -\langle\tau\,|^{\rm N}_{\rm N}|\,\xi_s'\rangle\end{pmatrix}\mathbf{U}(\alpha_s'),$$

where $\alpha_s = \frac{Q}{2}+\xi_s,\ \alpha_s' = \frac{Q}{2}+\xi_s',\ \xi_s,\xi_s'\in i\mathbb{R}_+$ and the symbols $\langle\tau\,|^{\rm N}_{\rm N}|\,\xi_s\rangle$ etc. are defined in subsection 5.2. Using (5.12), (5.13) and relations:

$$|S_{\rm NS}(2\alpha_s)|^2 = S_{\rm NS}(Q-2\xi_s)S_{\rm NS}(Q+2\xi_s) = \frac{S_{\rm NS}(2\xi_s+Q)}{S_{\rm NS}(2\xi_s)},$$

we thus get

$$\int_{\mathbb{R}} dx\ \Theta_s^\dagger(x|\alpha_s)\Theta_s(x|\alpha_s') = \sigma_0\ i\delta(\alpha_s-\alpha_s'). \tag{4.1}$$

Furthermore, since

$$\theta_{\rm NN}(x|\alpha_s)\overline{\theta_{\rm NN}(y|\alpha_s)} = \iint_{-i\infty}^{i\infty}\frac{d\tau}{i}\frac{d\lambda}{i}\ {\rm e}^{\pi x(Q+\tau-\alpha_1-\alpha_2)+\pi y(-\lambda+\alpha_1+\alpha_2-Q)}$$

$$\times\frac{S_{\rm NS}\left(\frac{Q}{2}+\tau+\alpha_1-\alpha_2\right)S_{\rm NS}\left(\frac{Q}{2}+\tau+\alpha_3-\bar\alpha_4\right)}{S_{\rm NS}\left(\frac{Q}{2}+\lambda+\alpha_1-\alpha_2\right)S_{\rm NS}\left(\frac{Q}{2}+\lambda+\alpha_3-\bar\alpha_4\right)}\ \langle\tau\,|^{\rm N}_{\rm N}|\,\xi_s\rangle\,\langle\xi_s\,|^{\rm N}_{\rm N}|\,\lambda\rangle\,,$$

and

$$\theta_{\rm NR}(x|\alpha_s)\overline{\theta_{\rm NR}(y|\alpha_s)} = \iint_{-i\infty}^{i\infty}\frac{d\tau}{i}\frac{d\lambda}{i}\ {\rm e}^{\pi x(Q+\tau-\alpha_1-\alpha_2)+\pi y(-\lambda+\alpha_1+\alpha_2-Q)}$$

$$\times\frac{S_{\rm NS}\left(\frac{Q}{2}+\tau+\alpha_1-\alpha_2\right)S_{\rm NS}\left(\frac{Q}{2}+\tau+\alpha_3-\bar\alpha_4\right)}{S_{\rm NS}\left(\frac{Q}{2}+\lambda+\alpha_1-\alpha_2\right)S_{\rm NS}\left(\frac{Q}{2}+\lambda+\alpha_3-\bar\alpha_4\right)}\ \langle\tau\,|^{\rm R}_{\rm R}|\,\xi_s\rangle\,\langle\xi_s\,|^{\rm R}_{\rm R}|\,\lambda\rangle\,,$$

we get

$$\int\limits_{\alpha_s\in\frac{Q}{2}+i\mathbb{R}_+} \frac{d\alpha_s}{8i}\;|S_{\rm NS}(2\alpha_s)|^2\;\left[\theta_{\rm NN}(x|\alpha_s)\overline{\theta}_{\rm NN}(y|\alpha_s)+\theta_{\rm NR}(x|\alpha_s)\overline{\theta}_{\rm NR}(y|\alpha_s)\right]$$

$$= \iint\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\frac{d\lambda}{i}\;{\rm e}^{\pi x(Q+\tau-\alpha_1-\alpha_2)+\pi y(-\lambda+\alpha_1+\alpha_2-Q)}$$

$$\times\frac{S_{\rm NS}(\frac{Q}{2}+\tau+\alpha_1-\alpha_2)S_{\rm NS}(\frac{Q}{2}+\tau+\alpha_3-\bar{\alpha}_4)}{S_{\rm NS}(\frac{Q}{2}+\lambda+\alpha_1-\alpha_2)S_{\rm NS}(\frac{Q}{2}+\lambda+\alpha_3-\bar{\alpha}_4)}$$

$$\times\int\limits_{-i\infty}^{i\infty}\frac{d\xi_s}{16i}\;\nu(\xi_s)\Big(\langle\tau\,|^{\rm N}_{\rm N}|\,\xi_s\rangle\,\langle\xi_s\,|^{\rm N}_{\rm N}|\,\lambda\rangle+\langle\tau\,|^{\rm R}_{\rm R}|\,\xi_s\rangle\,\langle\xi_s\,|^{\rm R}_{\rm R}|\,\lambda\rangle\Big)\;=\;\delta(x-y),$$

where we used the symmetry $\xi_s\to-\xi_s$ of the function

$$\nu(\xi_s)\Big(\langle\tau\,|^{\rm N}_{\rm N}|\,\xi_s\rangle\,\langle\xi_s\,|^{\rm N}_{\rm N}|\,\lambda\rangle+\langle\tau\,|^{\rm R}_{\rm R}|\,\xi_s\rangle\,\langle\xi_s\,|^{\rm R}_{\rm R}|\,\lambda\rangle\Big)$$

to extend the ξ_s integration over the entire imaginary axis and applied eq. (5.15). Repeating essentially the same calculation (and using eq. (5.16) for the off-diagonal elements) we eventually get

$$\int\limits_{\alpha_s\in\frac{Q}{2}+i\mathbb{R}_+}\frac{d\alpha_s}{i}\;\Theta_s(x|\alpha_s)\Theta_s^\dagger(y|\alpha_s)=\sigma_0\,\delta(x-y). \tag{4.2}$$

Analogous orthogonality and completeness relations are satisfied by $\Theta_t(x|\alpha_t)$,

$$\int\limits_{\mathbb{R}}dx\;\Theta_t^\dagger(x|\alpha_t)\Theta_t(x|\alpha_t')=\sigma_0\;i\delta(\alpha_t-\alpha_t'), \tag{4.3}$$

and

$$\int\limits_{\alpha_t\in\frac{Q}{2}+i\mathbb{R}_+}\frac{d\alpha_t}{i}\;\Theta_t(x|\alpha_t)\Theta_t^\dagger(y|\alpha_t)=\sigma_0\,\delta(x-y). \tag{4.4}$$

Consequently:

$$\int\limits_{\frac{Q}{2}+i\mathbb{R}_+}\frac{d\alpha_s}{i}\;|S_{\rm NS}(2\alpha_s)|^2\;\mathbf{G}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3&\alpha_2\\\alpha_4&\alpha_1\end{smallmatrix}\right]\mathbf{G}^\dagger_{\alpha_s\alpha_t'}\left[\begin{smallmatrix}\alpha_3&\alpha_2\\\alpha_4&\alpha_1\end{smallmatrix}\right]$$

$$= S_{\rm NS}(2\alpha_t)\overline{S_{\rm NS}(2\alpha_t')}$$

$$\times\left(\mathbf{M}^t_{\alpha_t}\left[\begin{smallmatrix}\alpha_3&\alpha_2\\\alpha_4&\alpha_1\end{smallmatrix}\right]\right)^\dagger\iint\limits_{\mathbb{R}}dxdy\;\Theta_t^\dagger(x|\alpha_t)\left(\;\int\limits_{\frac{Q}{2}+i\mathbb{R}_+}\frac{d\alpha_s}{i}\;\Theta_s(x|\alpha_s)\Theta_s^\dagger(y|\alpha_s)\right)\Theta_t(y|\alpha_t')\;\mathbf{M}^t_{\alpha_t'}\left[\begin{smallmatrix}\alpha_3&\alpha_2\\\alpha_4&\alpha_1\end{smallmatrix}\right]$$

$$= S_{\rm NS}(2\alpha_t)\overline{S_{\rm NS}(2\alpha_t')}\;\left(\mathbf{M}^t_{\alpha_t}\left[\begin{smallmatrix}\alpha_3&\alpha_2\\\alpha_4&\alpha_1\end{smallmatrix}\right]\right)^\dagger\int\limits_{\mathbb{R}}dx\;\Theta_t^\dagger(x|\alpha_t)\Theta_t(x|\alpha_t')\;\mathbf{M}^t_{\alpha_t'}\left[\begin{smallmatrix}\alpha_3&\alpha_2\\\alpha_4&\alpha_1\end{smallmatrix}\right]$$

$$= |S_{\rm NS}(2\alpha_t)|^2\,\sigma_0\;i\delta(\alpha_t-\alpha_t'),$$

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where we used (4.2), (4.3) and unitarity of $\mathbf{M}^{t}_{\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$. In result the equality (3.8) implies validity of the bootstrap equation for the four-point correlator of the NS super-primary fields.

4.2 Symmetry properties

Using (3.4) one can work out an explicit expression for the matrix $\mathbf{G}$. It is of the form:

$$
\begin{aligned}
\mathbf{G}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{\imath}{}_{\jmath} = {} & \frac{S_\imath(\alpha_t+\alpha_1-\bar{\alpha}_4)S_\imath(\alpha_t-\alpha_3+\bar{\alpha}_2)}{S_\imath(\alpha_t+\alpha_1-\alpha_4)S_\imath(\alpha_t-\alpha_3+\alpha_2)}\frac{S_\jmath(\alpha_s+\alpha_3-\alpha_4)S_\jmath(\alpha_s-\alpha_1+\alpha_2)}{S_\jmath(\alpha_s+\alpha_3-\bar{\alpha}_4)S_\jmath(\alpha_s-\alpha_1+\bar{\alpha}_2)}\\
&\times\frac{|S_{\mathrm{NS}}(2\alpha_t)|^2}{4}\int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\,\mathbf{J}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{\imath}{}_{\jmath},
\end{aligned}
\tag{4.5}
$$

where the superscript $\imath = 1$ (subscript $\jmath = 1$) on the l.h.s. corresponds to the function S_{NS} on the r.h.s, the superscript $\imath = 2$ (subscript $\jmath = 2$) on the l.h.s. corresponds to the function S_{R} on the r.h.s. and:

$$
\begin{aligned}
\mathbf{J}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{1}{}_{1} = {} & \frac{S_{\mathrm{NS}}(\tau+\bar{\alpha}_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\mathrm{NS}}(\tau+\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\bar{\alpha}_3)}{S_{\mathrm{NS}}(\tau+\alpha_4+\bar{\alpha}_t)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_t)S_{\mathrm{NS}}(\tau+\alpha_2+\bar{\alpha}_s)S_{\mathrm{NS}}(\tau+\alpha_2+\alpha_s)}\\
&+\frac{S_{\mathrm{R}}(\tau+\bar{\alpha}_1)S_{\mathrm{R}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\mathrm{R}}(\tau+\alpha_1)S_{\mathrm{R}}(\tau+\alpha_4+\alpha_2-\bar{\alpha}_3)}{S_{\mathrm{R}}(\tau+\alpha_4+\bar{\alpha}_t)S_{\mathrm{R}}(\tau+\alpha_4+\alpha_t)S_{\mathrm{R}}(\tau+\alpha_2+\bar{\alpha}_s)S_{\mathrm{R}}(\tau+\alpha_2+\alpha_s)},\\
\mathbf{J}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{1}{}_{2} = {} & \frac{S_{\mathrm{NS}}(\tau+\bar{\alpha}_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\mathrm{NS}}(\tau+\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\bar{\alpha}_3)}{S_{\mathrm{NS}}(\tau+\alpha_4+\bar{\alpha}_t)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_t)S_{\mathrm{R}}(\tau+\alpha_2+\bar{\alpha}_s)S_{\mathrm{R}}(\tau+\alpha_2+\alpha_s)}\\
&-\frac{S_{\mathrm{R}}(\tau+\bar{\alpha}_1)S_{\mathrm{R}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\mathrm{R}}(\tau+\alpha_1)S_{\mathrm{R}}(\tau+\alpha_4+\alpha_2-\bar{\alpha}_3)}{S_{\mathrm{R}}(\tau+\alpha_4+\bar{\alpha}_t)S_{\mathrm{R}}(\tau+\alpha_4+\alpha_t)S_{\mathrm{NS}}(\tau+\alpha_2+\bar{\alpha}_s)S_{\mathrm{NS}}(\tau+\alpha_2+\alpha_s)},\\
\mathbf{J}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{2}{}_{1} = {} & \frac{S_{\mathrm{NS}}(\tau+\bar{\alpha}_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\mathrm{NS}}(\tau+\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\bar{\alpha}_3)}{S_{\mathrm{R}}(\tau+\alpha_4+\bar{\alpha}_t)S_{\mathrm{R}}(\tau+\alpha_4+\alpha_t)S_{\mathrm{NS}}(\tau+\alpha_2+\bar{\alpha}_s)S_{\mathrm{NS}}(\tau+\alpha_2+\alpha_s)}\\
&-\frac{S_{\mathrm{R}}(\tau+\bar{\alpha}_1)S_{\mathrm{R}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\mathrm{R}}(\tau+\alpha_1)S_{\mathrm{R}}(\tau+\alpha_4+\alpha_2-\bar{\alpha}_3)}{S_{\mathrm{NS}}(\tau+\alpha_4+\bar{\alpha}_t)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_t)S_{\mathrm{R}}(\tau+\alpha_2+\bar{\alpha}_s)S_{\mathrm{R}}(\tau+\alpha_2+\alpha_s)},\\
\mathbf{J}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{2}{}_{2} = {} & \frac{S_{\mathrm{NS}}(\tau+\bar{\alpha}_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\mathrm{NS}}(\tau+\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\bar{\alpha}_3)}{S_{\mathrm{R}}(\tau+\alpha_4+\bar{\alpha}_t)S_{\mathrm{R}}(\tau+\alpha_4+\alpha_t)S_{\mathrm{R}}(\tau+\alpha_2+\bar{\alpha}_s)S_{\mathrm{R}}(\tau+\alpha_2+\alpha_s)}\\
&+\frac{S_{\mathrm{R}}(\tau+\bar{\alpha}_1)S_{\mathrm{R}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\mathrm{R}}(\tau+\alpha_1)S_{\mathrm{R}}(\tau+\alpha_4+\alpha_2-\bar{\alpha}_3)}{S_{\mathrm{NS}}(\tau+\alpha_4+\bar{\alpha}_t)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_t)S_{\mathrm{NS}}(\tau+\alpha_2+\bar{\alpha}_s)S_{\mathrm{NS}}(\tau+\alpha_2+\alpha_s)}.
\end{aligned}
$$

Multiplying elements of (4.5) with the corresponding elements of normalization factors N (see eq. (3.7)) we obtain an explicit expression for the matrix $\mathbf{F}$:

$$
\begin{aligned}
\mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{\imath}{}_{\jmath} = {} & \frac{\Gamma_\imath(\bar{\alpha}_t+\bar{\alpha}_3-\alpha_2)\Gamma_\imath(\bar{\alpha}_t+\alpha_3-\alpha_2)\Gamma_\imath(\alpha_t+\bar{\alpha}_3-\alpha_2)\Gamma_\imath(\alpha_t+\alpha_3-\alpha_2)}{\Gamma_\jmath(\bar{\alpha}_s+\bar{\alpha}_1-\alpha_2)\Gamma_\jmath(\bar{\alpha}_s+\alpha_1-\alpha_2)\Gamma_\jmath(\alpha_s+\bar{\alpha}_1-\alpha_2)\Gamma_\jmath(\alpha_s+\alpha_1-\alpha_2)}\\
&\times\frac{\Gamma_\imath(\bar{\alpha}_t+\bar{\alpha}_1-\bar{\alpha}_4)\Gamma_\imath(\bar{\alpha}_t+\alpha_1-\bar{\alpha}_4)\Gamma_\imath(\alpha_t+\bar{\alpha}_1-\bar{\alpha}_4)\Gamma_\imath(\alpha_t+\alpha_1-\bar{\alpha}_4)}{\Gamma_\jmath(\bar{\alpha}_s+\bar{\alpha}_3-\bar{\alpha}_4)\Gamma_\jmath(\bar{\alpha}_s+\alpha_3-\bar{\alpha}_4)\Gamma_\jmath(\alpha_s+\bar{\alpha}_3-\bar{\alpha}_4)\Gamma_\jmath(\alpha_s+\alpha_3-\bar{\alpha}_4)}\\
&\times\frac{\Gamma_{\mathrm{NS}}(2\alpha_s)\Gamma_{\mathrm{NS}}(2\bar{\alpha}_s)}{4\,\Gamma_{\mathrm{NS}}(\alpha_t-\bar{\alpha}_t)\Gamma_{\mathrm{NS}}(\bar{\alpha}_t-\alpha_t)}\int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\,\mathbf{J}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{\imath}{}_{\jmath}.
\end{aligned}
\tag{4.6}
$$

Notice that $\mathbf{F}$ is explicitly invariant with respect to conjugations $\alpha_i \to Q - \alpha_i$ for $i = s, t, 1, 3$. Employing (3.2) we further get:

$$\mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_2 & \alpha_3\\ \bar{\alpha}_1 & \bar{\alpha}_4\end{smallmatrix}\right].$$

The matrix on the r.h.s. of this equation is explicitly invariant with respect to conjugations $\alpha_2 \to Q - \alpha_2$ and $\alpha_4 \to Q - \alpha_4$. This property therefore holds also for the matrix on the l.h.s. The matrix $\mathbf{F}$ thus depends only on the conformal weights and enjoys an invariance with respect to exchange of its columns,

$$\mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_2 & \alpha_3\\ \alpha_1 & \alpha_4\end{smallmatrix}\right]. \tag{4.7}$$

Finally, since the deformed hypergeometric function (3.1) is invariant with respect to exchange of its first two arguments,

$$\mathbb{F}(\alpha, \beta; \gamma; z) = \mathbb{F}(\beta, \alpha; \gamma, z),$$

all the factors in the definition of the matrix $\mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ are invariant with respect to the simultaneous exchange $\alpha_1 \leftrightarrow \alpha_3$ and $\alpha_2 \leftrightarrow \bar{\alpha}_4$. We thus have

$$\mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_1 & \bar{\alpha}_4\\ \alpha_2 & \alpha_3\end{smallmatrix}\right] = \mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_1 & \alpha_4\\ \alpha_2 & \alpha_3\end{smallmatrix}\right],$$

and using (4.7) we finally conclude that the matrix $\mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ is invariant with respect to exchange of its rows as well,

$$\mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \mathbf{F}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_4 & \alpha_1\\ \alpha_3 & \alpha_2\end{smallmatrix}\right]. \tag{4.8}$$

4.3 Special values of the arguments

In this subsection we shall discuss the limit $\alpha_2 \to -b$. Four point correlation functions containing degenerate field V_{-b} satisfy a third order, ordinary, linear differential equation which can be derived from the requirement of decoupling of the null field $\left(L_{-1}S_{-\frac{1}{2}} + b^2 S_{-\frac{3}{2}}\right) V_{-b}$. It follows form the form of this equation that the operator product expansion of V_{-b} with an arbitrary primary field V_{α_1} decomposes onto three conformal families [16]:

$$V_{-b}\, V_{\alpha_1} \;\in\; [V_{\alpha_1 - b}]_{\rm ee} + [V_{\alpha_1}]_{\rm oo} + [V_{\alpha_1 + b}]_{\rm ee}\,. \tag{4.9}$$

This property is reflected by the fact that the conformal block $\mathcal{F}^{\rm e}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)$ possesses a well defined limit $\alpha_2 \to -b$ if and only if $\alpha_2 = \alpha_1 \pm b$, while $\lim\limits_{\alpha_2 \to -b} \mathcal{F}^{\rm o}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)$ exists if and only if $\alpha_s = \alpha_1$. The four-point correlation function of the field V_{-b} and the super-primary fields V_{α_1}, V_{α_3}, V_{α_4} with arbitrary $\alpha_1, \alpha_3, \alpha_4 \in \frac{Q}{2} + i\mathbb{R}_+$ can be thus expressed through these three blocks. Since this property cannot depend on our choice of the decomposition "channel", the integral in the equation (2.6) must descent for $\alpha_s = \alpha_1 \pm b, 0$ and $\alpha_2 \to -b$ to a sum containing just three terms.

Let us check this for the block $\mathcal{F}^{\rm e}_{\alpha_1+b}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)$. To this end we need to calculate the limits

$$\lim_{\alpha_2 \to 0} \mathbf{F}_{\alpha_1+b,\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{\imath}{}_1 = \lim_{\alpha_2 \to 0} \mathbf{F}_{\alpha_1+b,\alpha_t}\left[\begin{smallmatrix}\alpha_4 & \alpha_1\\ \alpha_3 & \alpha_2\end{smallmatrix}\right]^{\imath}{}_1, \qquad \imath = 1, 2. \tag{4.10}$$

For $\alpha_s = \alpha_1 - b$ the pre-factor of the integral in $\mathbf{F}_{\alpha_1+b,\alpha_t}\left[\begin{smallmatrix}\alpha_4 & \alpha_1\\ \alpha_3 & \alpha_2\end{smallmatrix}\right]^{\imath}{}_1$ contains a term $\Gamma_{\rm NS}^{-1}(\alpha_2+b)$, and thus (4.10) vanishes unless there is a compensating, singular factor coming form the integral in (4.6). As explained in ([9], Lemma 3), such a singular term can arise in the process of analytic continuation of the integral

$$\int_{-i\infty}^{i\infty} \frac{d\tau}{i}\, \mathbf{J}_{\alpha_1+b,\alpha_t}\left[\begin{smallmatrix}\alpha_4 & \alpha_1\\ \alpha_3 & \alpha_2\end{smallmatrix}\right]^{1}{}_1 = I_{\rm NS}(\alpha_2) + I_{\rm R}(\alpha_2),$$

$$I_{\rm NS}(\alpha_2) = \int_{-i\infty}^{i\infty} \frac{d\tau}{i}\, \frac{S_{\rm NS}(\tau+\bar\alpha_2)S_{\rm NS}(\tau+\alpha_3+\alpha_1-\alpha_4)S_{\rm NS}(\tau+\alpha_2)S_{\rm NS}(\tau+\alpha_3+\alpha_1-\bar\alpha_4)}{S_{\rm NS}(\tau+\alpha_3+\bar\alpha_t)S_{\rm NS}(\tau+\alpha_3+\alpha_t)S_{\rm NS}(\tau+Q-b)S_{\rm NS}(\tau+2\alpha_1+b)},$$

$$I_{\rm R}(\alpha_2) = \int_{-i\infty}^{i\infty} \frac{d\tau}{i}\, \frac{S_{\rm R}(\tau+\bar\alpha_2)S_{\rm R}(\tau+\alpha_3+\alpha_1-\alpha_4)S_{\rm R}(\tau+\alpha_2)S_{\rm R}(\tau+\alpha_3+\alpha_1-\bar\alpha_4)}{S_{\rm R}(\tau+\alpha_3+\bar\alpha_t)S_{\rm R}(\tau+\alpha_3+\alpha_t)S_{\rm R}(\tau+Q-b)S_{\rm R}(\tau+2\alpha_1+b)},$$

to $\alpha_2 = -b$ if the integration contour gets "pinched" between two poles of the integrand; the singular contribution is obtained by calculating the residue at any one of these poles. Such a pinching occurs in $I_{\rm NS}(\alpha_2)$, where the pole at $\tau = b$, coming from zero of the function $S_{\rm NS}(\tau+Q-b)$ in the denominator and located to the right of the contour, "collides" in the limit $\alpha_2 \to -b$ with a pole at $\tau = -\alpha_2$ from the term $S_{\rm NS}(\tau+\alpha_2)$ in the numerator, located to the left of the contour. Calculating the residue we get:

$$I^{(0)}_{\rm NS}(\alpha_2) = 2\frac{S_{\rm NS}(Q-2\alpha_2)S_{\rm NS}(\alpha_1+\alpha_3-\alpha_4-\alpha_2)S_{\rm NS}(\alpha_1+\alpha_3-\bar\alpha_4-\alpha_2)}{S_{\rm NS}(Q-b-\alpha_2)S_{\rm NS}(2\alpha_1+b-\alpha_2)S_{\rm NS}(\alpha_3+\alpha_t-\alpha_2)S_{\rm NS}(\alpha_3+\bar\alpha_t-\alpha_2)},$$

and consequently:

$$\begin{aligned}\lim_{\alpha_2\to -b} \mathbf{F}_{\alpha_1+b,\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{1}{}_1 = {}& \frac{\Gamma_{\rm NS}(2\bar\alpha_1-2b)}{\Gamma_{\rm NS}(2\bar\alpha_1)}\frac{\Gamma_{\rm NS}(\bar\alpha_4+\bar\alpha_t-\alpha_1)\Gamma_{\rm NS}(\bar\alpha_4+\alpha_t-\alpha_1)}{\Gamma_{\rm NS}(\bar\alpha_4+\bar\alpha_3-\alpha_1-b)\Gamma_{\rm NS}(\bar\alpha_4+\alpha_3-\alpha_1-b)}\\ &\times\frac{\Gamma_{\rm NS}(\alpha_4+\bar\alpha_t-\alpha_1)\Gamma_{\rm NS}(\alpha_4+\alpha_t-\alpha_1)}{\Gamma_{\rm NS}(\alpha_4+\bar\alpha_3-\alpha_1-b)\Gamma_{\rm NS}(\alpha_4+\alpha_3-\alpha_1-b)}\\ &\times\lim_{\alpha_2\to 0}\frac{\Gamma_{\rm NS}(\alpha_3+\alpha_2-\alpha_t)\Gamma_{\rm NS}(\alpha_t-\alpha_3+\alpha_2)\Gamma_{\rm NS}(\alpha_3+\alpha_2-\bar\alpha_t)\Gamma_{\rm NS}(\bar\alpha_t-\alpha_3+\alpha_2)}{2\,\Gamma_{\rm NS}(Q)\Gamma_{\rm NS}(2\alpha_2)\Gamma_{\rm NS}(2\alpha_t-Q)\Gamma_{\rm NS}(Q-2\alpha_t)}.\end{aligned}$$

Functions in the last line possess poles which move as we change α_2, pinching the contour of integration over α_t (which is originally the semi-line $\frac{Q}{2}+i\mathbb{R}_+$). We get two pairs of colliding poles: the poles of $\Gamma_{\rm NS}(\alpha_t-\alpha_3+\alpha_2)$ at $\alpha_t = \alpha_3-\alpha_2$ and $\alpha_3-\alpha_2-2b$ (to the left of the contour) collide with the poles of $\Gamma_{\rm NS}(\alpha_3+\alpha_2-\alpha_t)$ at $\alpha_t = \alpha_3+\alpha_2+2b$ and $\alpha_t = \alpha_3+\alpha_2$ (to the right of the contour). Calculating residues of the colliding poles we get:

$$\lim_{\alpha_2\to -b}\int_{\frac{Q}{2}+i\mathbb{R}_+}\frac{d\alpha_t}{i}\,\mathcal{F}^e_{\alpha_t}\left[\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_4 & \alpha_3\end{smallmatrix}\right](1-x)\,\mathbf{F}_{\alpha_1+b,\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{1}{}_1 = \sum_{s=\pm 1}\mathbf{F}_{+,s}\begin{bmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{bmatrix}\mathcal{F}^e_{\alpha_3+s\,b}\begin{bmatrix}\alpha_1 & -b\\ \alpha_4 & \alpha_3\end{bmatrix}(1-x),$$

where

$$\begin{aligned}\mathbf{F}_{+,+}\begin{bmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{bmatrix} = {}& \frac{\Gamma_{\rm NS}(2\bar\alpha_1-2b)}{\Gamma_{\rm NS}(2\bar\alpha_1)}\frac{\Gamma_{\rm NS}(Q-2\alpha_3)}{\Gamma_{\rm NS}(Q-2\alpha_3+2b)}\\ &\times\frac{\Gamma_{\rm NS}(\alpha_4+\alpha_3-\alpha_1+b)\Gamma_{\rm NS}(\bar\alpha_4+\alpha_3-\alpha_1+b)}{\Gamma_{\rm NS}(\alpha_4+\alpha_3-\alpha_1-b)\Gamma_{\rm NS}(\bar\alpha_4+\alpha_3-\alpha_1-b)},\end{aligned} \tag{4.11}$$

and

$$\mathbf{F}_{+,-}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \frac{\Gamma_{\rm NS}(2\bar\alpha_1-2b)}{\Gamma_{\rm NS}(2\bar\alpha_1)}\frac{\Gamma_{\rm NS}(2\alpha_3-Q)}{\Gamma_{\rm NS}(2\alpha_3-Q+2b)} \times\frac{\Gamma_{\rm NS}(\alpha_4+\bar\alpha_3-\alpha_1+b)\Gamma_{\rm NS}(\bar\alpha_4+\bar\alpha_3-\alpha_1+b)}{\Gamma_{\rm NS}(\alpha_4+\bar\alpha_3-\alpha_1-b)\Gamma_{\rm NS}(\bar\alpha_4+\bar\alpha_3-\alpha_1-b)}. \tag{4.12}$$

The case

$$\lim_{\alpha_2\to 0}\mathbf{F}_{\alpha_1+b,\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^2{}_1 = \lim_{\alpha_2\to 0}\mathbf{F}_{\alpha_1+b,\alpha_t}\left[\begin{smallmatrix}\alpha_4 & \alpha_1\\ \alpha_3 & \alpha_2\end{smallmatrix}\right]^2{}_1$$

is similar. Calculating the residue of the colliding pole we get:

$$\int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\;\mathbf{J}_{\alpha_1+b,\alpha_t}\left[\begin{smallmatrix}\alpha_4 & \alpha_1\\ \alpha_3 & \alpha_2\end{smallmatrix}\right]^2{}_1$$
$$= 2\frac{S_{\rm NS}(Q-2\alpha_2)S_{\rm NS}(\alpha_1+\alpha_3-\alpha_4-\alpha_2)S_{\rm NS}(\alpha_1+\alpha_3-\bar\alpha_4-\alpha_2)}{S_{\rm NS}(Q-b-\alpha_2)S_{\rm NS}(2\alpha_1+b-\alpha_2)S_{\rm R}(\alpha_3+\alpha_t-\alpha_2)S_{\rm R}(\alpha_3+\bar\alpha_t-\alpha_2)} + \text{ regular},$$

and

$$\lim_{\alpha_2\to -b}\mathbf{F}_{\alpha_1+b,\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^2{}_1 = \frac{\Gamma_{\rm NS}(2\bar\alpha_1-2b)}{\Gamma_{\rm NS}(2\bar\alpha_1)}\frac{\Gamma_{\rm R}(\bar\alpha_4+\bar\alpha_t-\alpha_1)\Gamma_{\rm R}(\bar\alpha_4+\alpha_t-\alpha_1)}{\Gamma_{\rm NS}(\bar\alpha_4+\bar\alpha_3-\alpha_1-b)\Gamma_{\rm NS}(\bar\alpha_4+\alpha_3-\alpha_1-b)}$$
$$\times\frac{\Gamma_{\rm R}(\alpha_4+\bar\alpha_t-\alpha_1)\Gamma_{\rm R}(\alpha_4+\alpha_t-\alpha_1)}{\Gamma_{\rm NS}(\alpha_4+\bar\alpha_3-\alpha_1-b)\Gamma_{\rm NS}(\alpha_4+\alpha_3-\alpha_1-b)}$$
$$\times\lim_{\alpha_2\to 0}\frac{\Gamma_{\rm R}(\alpha_3+\alpha_2-\alpha_t)\Gamma_{\rm R}(\alpha_t-\alpha_3+\alpha_2)\Gamma_{\rm R}(\alpha_3+\alpha_2-\bar\alpha_t)\Gamma_{\rm R}(\bar\alpha_t-\alpha_3+\alpha_2)}{2\,\Gamma_{\rm NS}(Q)\Gamma_{\rm NS}(2\alpha_2)\Gamma_{\rm NS}(2\alpha_t-Q)\Gamma_{\rm NS}(Q-2\alpha_t)}.$$

This time in the last line there is only one pair of poles pinching the α_t contour: a pole of the function $\Gamma_{\rm R}(\alpha_t-\alpha_3+\alpha_2)$ located at $\alpha_t=\alpha_3-\alpha_2-b$ collides in the limit $\alpha_2\to -b$ with a pole of the function $\Gamma_{\rm R}(\alpha_3+\alpha_2-\alpha_t)$ at $\alpha_t=\alpha_3+\alpha_2+b$. Calculating the residue we get:

$$\lim_{\alpha_2\to -b}\int\limits_{\frac{Q}{2}+i\mathbb{R}_+}\frac{d\alpha_t}{i}\;\mathcal{F}^o_{\alpha_t}\left[\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_4 & \alpha_3\end{smallmatrix}\right](1-x)\,\mathbf{F}_{\alpha_1+b,\alpha_3}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^2{}_1 = \mathbf{F}_{+,0}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\mathcal{F}^o_{\alpha_3}\left[\begin{smallmatrix}\alpha_1 & -b\\ \alpha_4 & \alpha_3\end{smallmatrix}\right](1-x),$$

where

$$\mathbf{F}_{+,0}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = c_0\,\frac{\Gamma_{\rm NS}(2\bar\alpha_1-2b)}{\Gamma_{\rm NS}(2\bar\alpha_1)}\frac{\Gamma_{\rm R}(2\alpha_3-Q-b)\Gamma_{\rm R}(Q-2\alpha_3-b)}{\Gamma_{\rm NS}(2\alpha_3-Q)\Gamma_{\rm NS}(Q-2\alpha_3)}\times$$
$$\times\frac{\Gamma_{\rm R}(\bar\alpha_4+\bar\alpha_3-\alpha_1)\Gamma_{\rm R}(\bar\alpha_4+\alpha_3-\alpha_1)}{\Gamma_{\rm NS}(\bar\alpha_4+\bar\alpha_3-\alpha_1-b)\Gamma_{\rm NS}(\bar\alpha_4+\alpha_3-\alpha_1-b)}$$
$$\times\frac{\Gamma_{\rm R}(\alpha_4+\bar\alpha_3-\alpha_1)\Gamma_{\rm R}(\alpha_4+\alpha_3-\alpha_1)}{\Gamma_{\rm NS}(\alpha_4+\bar\alpha_3-\alpha_1-b)\Gamma_{\rm NS}(\alpha_4+\alpha_3-\alpha_1-b)},$$

and where

$$c_0 = \lim_{\alpha_2\to -b}\frac{\Gamma_{\rm R}(\alpha_2)\Gamma_{\rm R}(2\alpha_2+b)}{\Gamma_{\rm NS}(\alpha_2+b)\Gamma_{\rm NS}(2\alpha_2)} = 2\cos\left(\frac{bQ}{2}\right)\frac{\Gamma(bQ)}{\Gamma\left(\frac{bQ}{2}\right)}\,b^{-\frac{bQ}{2}}.$$

One can check in the same way that the fusion equations for $\mathcal{F}^o_{\alpha_1}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)$ as well as for $\mathcal{F}^e_{\alpha_1-b}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)$ also contain only three terms, proportional to $\mathcal{F}^e_{\alpha_3\pm b}\left[\begin{smallmatrix}\alpha_1 & -b\\ \alpha_4 & \alpha_3\end{smallmatrix}\right](1-z)$ and

$\mathcal{F}^{o}_{\alpha_3}\left[\begin{smallmatrix}\alpha_1 & -b\\ \alpha_4 & \alpha_3\end{smallmatrix}\right](1-z)$ (the algebra involved becomes more elaborate — in the case of $\mathcal{F}^{o}_{\alpha_1}$ one gets from the $\mathbf{J}$ integral two contributions from the colliding poles and in the case of $\mathcal{F}^{e}_{\alpha_1-b}$ we have three pair of poles pinching the τ contour — and we shall not present these calculations here).

As a final check let us calculate for (4.12), (4.11) and (4.13) the corresponding elements of the $\mathbf{G}$ matrix (3.6). Using (3.7), (2.8) and (A.4) we get:

$$\begin{aligned}\mathbf{G}_{+,-}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] &= \frac{S_{\rm NS}(2\alpha_3-2b)}{S_{\rm NS}(2\alpha_3)}\frac{S_{\rm NS}(\alpha_1+\alpha_3-\alpha_4+b)}{S_{\rm NS}(\alpha_1+\alpha_3-\alpha_4-b)}\\ &= \frac{\cos\frac{\pi b}{2}(\alpha_1+\alpha_3-\alpha_4)\,\sin\frac{\pi b}{2}(\alpha_1+\alpha_3-\alpha_4-b)}{\cos\pi b\big(\alpha_3-\frac{b}{2}\big)\,\sin\pi b(\alpha_3-b)}, \qquad (4.13)\\ \mathbf{G}_{+,+}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] &= \frac{S_{\rm NS}(2\alpha_3-Q)}{S_{\rm NS}(2\alpha_3-Q+2b)}\frac{S_{\rm NS}(\alpha_1+\alpha_4-\alpha_3+b)}{S_{\rm NS}(\alpha_1+\alpha_4-\alpha_3-b)}\\ &= -\frac{\cos\frac{\pi b}{2}(\alpha_1+\alpha_4-\alpha_3)\,\sin\frac{\pi b}{2}(\alpha_1+\alpha_4-\alpha_3-b)}{\cos\pi b\big(\alpha_3-\frac{b}{2}\big)\,\sin\pi b\alpha_3}, \qquad (4.14)\end{aligned}$$

and

$$\begin{aligned}\mathbf{G}_{+,0}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] &= \frac{S_{\rm NS}(2\alpha_3)S_{\rm NS}(2\bar{\alpha}_3)}{S_{\rm R}(2\alpha_3+b)S_{\rm R}(2\bar{\alpha}_3+b)}\frac{S_{\rm NS}(\alpha_1+\alpha_3-\alpha_4+b)S_{\rm NS}(\alpha_1+\alpha_4-\alpha_3+b)}{S_{\rm R}(\alpha_1+\alpha_3-\alpha_4)S_{\rm R}(\alpha_1+\alpha_4-\alpha_3)}\\ &= \frac{\cos\frac{\pi b}{2}(\alpha_1+\alpha_3-\alpha_4)\cos\frac{\pi b}{2}(\alpha_1+\alpha_4-\alpha_3)}{\sin\pi b\alpha_3\,\sin\pi b(\alpha_3-b)}. \qquad (4.15)\end{aligned}$$

It is reassuring to compare (4.13), (4.15) with the results of [16], where $\mathcal{F}^{e}_{\alpha_1\pm b}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)$ and $\mathcal{F}^{o}_{\alpha_1}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)$ have been calculated (in the double integral representation of Dotsenko and Fateev [20]) by solving the corresponding null vector decoupling equation (see especially appendix C of [16]).

Let us conclude this section with the following comment.[4] As in the case of the conformal blocks, application the Moore-Seiberg formalism [7] results in a set of a consistency condition for the fusion matrices of the superconformal blocks: the pentagon and the hexagon equations. In the appropriate limit these equations descend to a set of linear and quadratic equations for a generic fusion matrix elements with $\mathsf{F}_{r,s}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ as (some of the) coefficients.[5] This set of equations (together with the self-duality $b\to b^{-1}$ and the symmetry properties discussed in the subsection 4.2) is expected to characterize the fusion matrix uniquely. In view of this fact the coincidence of coefficients calculated in this section and the special fusion coefficients calculated in [16],

$$\mathbf{F}_{+,s}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \mathsf{F}_{+,s}\left[\begin{smallmatrix}\alpha_3 & -b\\ \alpha_4 & \alpha_1\end{smallmatrix}\right], \qquad s=\pm,0,$$

is a rather strong argument for the validity of the relation (3.8).

[4] I would like to thank the referee for an encouragement to discuss this point.

[5] The other coefficients are given by the special values of the fusion matrices of the blocks $\mathcal{F}_\alpha\left[\begin{smallmatrix}\alpha_3 & {}^*\alpha_2\\ \alpha_4 & {}^*\alpha_1\end{smallmatrix}\right](z)$ and $\mathcal{F}_\alpha\left[\begin{smallmatrix}{}^*\alpha_3 & {}^*\alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)$, see [14] for the notation.

5. Special functions related to the Barnes gamma

5.1 Integral identities

We shall derive several identities satisfied by a hypergeometric-type integrals constructed out of the Barnes functions $G_{\rm NS,R}(x)$ and $S_{\rm NS,R}(x)$, generalizing results of [22]. They include analogs (for the motivation of this terminology see [23]) of the Ramanujan summation formula (equation (5.5)), ${}_1F_2$ Heine transformation (equation (5.6)), Euler-Heine transformation (equation (5.7)), and the Saalschütz summation formula (equations (5.8) and (5.9)).

Let us denote:

$$\begin{aligned} B^{(\pm)}_{\rm NS}(\alpha,\beta) &= \int\limits_{-i\infty}^{i\infty} \frac{d\tau}{i}\ {\rm e}^{i\pi\beta}\left[\frac{G_{\rm NS}(\tau+\alpha)}{G_{\rm NS}(\tau+Q-0^+)} \pm \frac{G_{\rm R}(\tau+\alpha)}{G_{\rm R}(\tau+Q)}\right], \\ B^{(\pm)}_{\rm R}(\alpha,\beta) &= \int\limits_{-i\infty}^{i\infty} \frac{d\tau}{i}\ {\rm e}^{i\pi\beta}\left[\frac{G_{\rm R}(\tau+\alpha)}{G_{\rm NS}(\tau+Q-0^+)} \pm \frac{G_{\rm NS}(\tau+\alpha)}{G_{\rm R}(\tau+Q)}\right]. \end{aligned} \tag{5.1}$$

Using (A.4) one obtains:

$$\begin{aligned} \left(1+{\rm e}^{i\pi b(\alpha+\beta)}\right) B^{(+)}_{\rm NS}(\alpha+b,\beta) &= \left(1+{\rm e}^{i\pi b\alpha}\right) B^{(+)}_{\rm R}(\alpha,\beta), \\ \left(1-{\rm e}^{i\pi b(\alpha+\beta)}\right) B^{(+)}_{\rm R}(\alpha+b,\beta) &= \left(1-{\rm e}^{i\pi b\alpha}\right) B^{(+)}_{\rm NS}(\alpha,\beta), \\ \left(1-{\rm e}^{i\pi b(\alpha+\beta)}\right) B^{(-)}_{\rm NS}(\alpha+b,\beta) &= \left(1+{\rm e}^{i\pi b\alpha}\right) B^{(-)}_{\rm R}(\alpha,\beta), \\ \left(1+{\rm e}^{i\pi b(\alpha+\beta)}\right) B^{(-)}_{\rm R}(\alpha+b,\beta) &= \left(1-{\rm e}^{i\pi b\alpha}\right) B^{(-)}_{\rm NS}(\alpha,\beta), \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \left(1+{\rm e}^{i\pi b(\alpha+\beta)}\right) B^{(+)}_{\rm NS}(\alpha,\beta+b) &= \left(1+{\rm e}^{i\pi b\beta}\right) B^{(-)}_{\rm NS}(\alpha,\beta), \\ \left(1-{\rm e}^{i\pi b(\alpha+\beta)}\right) B^{(-)}_{\rm NS}(\alpha,\beta+b) &= \left(1-{\rm e}^{i\pi b\beta}\right) B^{(+)}_{\rm NS}(\alpha,\beta), \\ \left(1-{\rm e}^{i\pi b(\alpha+\beta)}\right) B^{(+)}_{\rm R}(\alpha,\beta+b) &= \left(1+{\rm e}^{i\pi b\beta}\right) B^{(-)}_{\rm R}(\alpha,\beta), \\ \left(1+{\rm e}^{i\pi b(\alpha+\beta)}\right) B^{(-)}_{\rm R}(\alpha,\beta+b) &= \left(1-{\rm e}^{i\pi b\beta}\right) B^{(+)}_{\rm R}(\alpha,\beta). \end{aligned} \tag{5.3}$$

For $b \in \mathbb{R}\setminus\mathbb{Q}$ equations (5.2), (5.3), together with their counterparts obtained by substituting $b \to b^{-1}$, determine the functions involved up to α and β independent factors,

$$\begin{aligned} B^{(+)}_{\rm NS}(\alpha,\beta) &= C^{(+)}_{\rm NS}\,\frac{G_{\rm NS}(\alpha)G_{\rm NS}(\beta)}{G_{\rm NS}(\alpha+\beta)}, & B^{(-)}_{\rm NS}(\alpha,\beta) &= C^{(-)}_{\rm NS}\,\frac{G_{\rm NS}(\alpha)G_{\rm R}(\beta)}{G_{\rm R}(\alpha+\beta)}, \\ B^{(+)}_{\rm R}(\alpha,\beta) &= C^{(+)}_{\rm R}\,\frac{G_{\rm R}(\alpha)G_{\rm NS}(\beta)}{G_{\rm R}(\alpha+\beta)}, & B^{(-)}_{\rm R}(\alpha,\beta) &= C^{(-)}_{\rm R}\,\frac{G_{\rm R}(\alpha)G_{\rm R}(\beta)}{G_{\rm NS}(\alpha+\beta)}. \end{aligned} \tag{5.4}$$

The factors $C^{(\pm)}_{\rm NS,R}$ can be computed by explicitly calculating all the functions appearing in (5.4) at special values of α; since:

$$B^{(\pm)}_{\rm R}\left(b^{-1},\beta\right) = \int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\;{\rm e}^{i\pi\tau\beta}\left[\frac{1}{1-{\rm e}^{i\pi b\tau}}\pm\frac{1}{1+{\rm e}^{i\pi b\tau}}\right] = \frac{2}{ib(1\mp {\rm e}^{i\pi b^{-1}\beta})},$$

and

$$B^{(+)}_{\rm NS}\left(b^{-1}-b,\beta\right) = \int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\frac{{\rm e}^{i\pi\tau\beta}}{\left(1-{\rm e}^{i\pi b\tau}\right)\left(1+q^{-1}{\rm e}^{i\pi b\tau}\right)} = \frac{2}{ib(1+q)}\frac{q+{\rm e}^{i\pi Q\beta}}{1-{\rm e}^{2\pi i b^{-1}\beta}},$$

$$B^{(-)}_{\rm NS}\left(b^{-1}-b,\beta\right) = \int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\frac{{\rm e}^{i\pi\tau\beta}}{\left(1+{\rm e}^{i\pi b\tau}\right)\left(1-q^{-1}{\rm e}^{i\pi b\tau}\right)} = \frac{2}{ib(1+q)}\frac{q{\rm e}^{i\pi b^{-1}\beta}+{\rm e}^{i\pi b\beta}}{1-{\rm e}^{2\pi i b^{-1}\beta}}$$

with $q={\rm e}^{i\pi b^2}$, we get a "supersymmetric", integral analog of the Ramanujan summation formula:

$$\begin{aligned}
\int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\;{\rm e}^{i\pi\tau\beta}\frac{G_{\rm NS}(\tau+\alpha)}{G_{\rm NS}(\tau+Q-0^+)} &= {\rm e}^{\frac{i\pi Q^2}{8}}G_{\rm NS}(\alpha)\left[\frac{G_{\rm NS}(\beta)}{G_{\rm NS}(\alpha+\beta)}+\frac{G_{\rm R}(\beta)}{G_{\rm R}(\alpha+\beta)}\right],\\
\int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\;{\rm e}^{i\pi\tau\beta}\frac{G_{\rm R}(\tau+\alpha)}{G_{\rm R}(\tau+Q)} &= {\rm e}^{\frac{i\pi Q^2}{8}}G_{\rm NS}(\alpha)\left[\frac{G_{\rm NS}(\beta)}{G_{\rm NS}(\alpha+\beta)}-\frac{G_{\rm R}(\beta)}{G_{\rm R}(\alpha+\beta)}\right],\\
\int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\;{\rm e}^{i\pi\tau\beta}\frac{G_{\rm R}(\tau+\alpha)}{G_{\rm NS}(\tau+Q-0^+)} &= {\rm e}^{\frac{i\pi Q^2}{8}}G_{\rm R}(\alpha)\left[\frac{G_{\rm NS}(\beta)}{G_{\rm R}(\alpha+\beta)}+\frac{G_{\rm R}(\beta)}{G_{\rm NS}(\alpha+\beta)}\right],\\
\int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\;{\rm e}^{i\pi\tau\beta}\frac{G_{\rm NS}(\tau+\alpha)}{G_{\rm R}(\tau+Q)} &= {\rm e}^{\frac{i\pi Q^2}{8}}G_{\rm R}(\alpha)\left[\frac{G_{\rm NS}(\beta)}{G_{\rm R}(\alpha+\beta)}-\frac{G_{\rm R}(\beta)}{G_{\rm NS}(\alpha+\beta)}\right].
\end{aligned}\tag{5.5}$$

Formulae (5.5) allow to derive an analog of the ${}_1F_2$ Heine transformation. Let us introduce a shorthand notation:

$$\begin{bmatrix}\rm NN\\ \rm NN\end{bmatrix}(\alpha,\beta;\gamma;z) = \int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\;{\rm e}^{i\pi\tau z}\frac{G_{\rm NS}(\tau+\alpha)G_{\rm NS}(\tau+\beta)}{G_{\rm NS}(\tau+\gamma-0^+)G_{\rm NS}(\tau+Q-0^+)},$$

$$\begin{bmatrix}\rm RR\\ \rm NN\end{bmatrix}(\alpha,\beta;\gamma;z) = \int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\;{\rm e}^{i\pi\tau z}\frac{G_{\rm R}(\tau+\alpha)G_{\rm R}(\tau+\beta)}{G_{\rm NS}(\tau+\gamma-0^+)G_{\rm NS}(\tau+Q-0^+)},$$

$$\vdots$$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}_{\rm NS} = \frac{G_{\rm NS}(\alpha)}{G_{\rm NS}(\beta)},\qquad \begin{pmatrix}\alpha\\ \beta\end{pmatrix}_{\rm R} = \frac{G_{\rm R}(\alpha)}{G_{\rm R}(\beta)}.$$

Using (5.5) to express a ratio of the first function from the numerator and the first function from the denominator as an integral, changing the order of integration and using (5.5) again we get:

$$\begin{aligned}
\left(\begin{bmatrix}\mathrm{NN}\\\mathrm{NN}\end{bmatrix}+\begin{bmatrix}\mathrm{RR}\\\mathrm{RR}\end{bmatrix}\right)(\alpha,\beta;\gamma;z) &= \begin{pmatrix}\beta\\\gamma-\alpha\end{pmatrix}_{\mathrm{NS}}\left(\begin{bmatrix}\mathrm{NN}\\\mathrm{NN}\end{bmatrix}+\begin{bmatrix}\mathrm{RR}\\\mathrm{RR}\end{bmatrix}\right)(z,\gamma-\alpha;z+\beta;\alpha),\\
\left(\begin{bmatrix}\mathrm{NN}\\\mathrm{NN}\end{bmatrix}-\begin{bmatrix}\mathrm{RR}\\\mathrm{RR}\end{bmatrix}\right)(\alpha,\beta;\gamma;z) &= \begin{pmatrix}\beta\\\gamma-\alpha\end{pmatrix}_{\mathrm{NS}}\left(\begin{bmatrix}\mathrm{RN}\\\mathrm{RN}\end{bmatrix}+\begin{bmatrix}\mathrm{NR}\\\mathrm{NR}\end{bmatrix}\right)(z,\gamma-\alpha;z+\beta;\alpha),\\
\left(\begin{bmatrix}\mathrm{RN}\\\mathrm{RN}\end{bmatrix}-\begin{bmatrix}\mathrm{NR}\\\mathrm{NR}\end{bmatrix}\right)(\alpha,\beta;\gamma;z) &= \begin{pmatrix}\beta\\\gamma-\alpha\end{pmatrix}_{\mathrm{NS}}\left(\begin{bmatrix}\mathrm{RN}\\\mathrm{RN}\end{bmatrix}-\begin{bmatrix}\mathrm{NR}\\\mathrm{NR}\end{bmatrix}\right)(z,\gamma-\alpha;z+\beta;\alpha),\\
\left(\begin{bmatrix}\mathrm{RN}\\\mathrm{RN}\end{bmatrix}+\begin{bmatrix}\mathrm{NR}\\\mathrm{NR}\end{bmatrix}\right)(\alpha,\beta;\gamma;z) &= \begin{pmatrix}\beta\\\gamma-\alpha\end{pmatrix}_{\mathrm{NS}}\left(\begin{bmatrix}\mathrm{NN}\\\mathrm{NN}\end{bmatrix}-\begin{bmatrix}\mathrm{RR}\\\mathrm{RR}\end{bmatrix}\right)(z,\gamma-\alpha;z+\beta;\alpha),
\end{aligned} \tag{5.6}$$

plus twelve similar formulae with the other combinations of the $G_{\mathrm{NS,R}}$ functions.

Combining (three times) formulae (5.6) with an exchange of the first two arguments of the involved functions one arrives at the "supersymmetric" integral analogues of the Euler-Heine transformations. Four out of sixteen formulae of this type read:

$$\begin{aligned}
&\left(\begin{bmatrix}\mathrm{NN}\\\mathrm{NN}\end{bmatrix}+\begin{bmatrix}\mathrm{RR}\\\mathrm{RR}\end{bmatrix}\right)(\alpha,\beta;\gamma;z)\\
&\quad= \begin{pmatrix}\alpha\\\gamma-\beta\end{pmatrix}_{\mathrm{NS}}\begin{pmatrix}z\\z-A\end{pmatrix}_{\mathrm{NS}}\begin{pmatrix}\beta\\\gamma-\alpha\end{pmatrix}_{\mathrm{NS}}\left(\begin{bmatrix}\mathrm{NN}\\\mathrm{NN}\end{bmatrix}+\begin{bmatrix}\mathrm{RR}\\\mathrm{RR}\end{bmatrix}\right)(\gamma-\beta,\gamma-\alpha;\gamma;z-A),\\
&\left(\begin{bmatrix}\mathrm{NN}\\\mathrm{NN}\end{bmatrix}-\begin{bmatrix}\mathrm{RR}\\\mathrm{RR}\end{bmatrix}\right)(\alpha,\beta;\gamma;z)\\
&\quad= \begin{pmatrix}\alpha\\\gamma-\beta\end{pmatrix}_{\mathrm{NS}}\begin{pmatrix}z\\z-A\end{pmatrix}_{\mathrm{R}}\begin{pmatrix}\beta\\\gamma-\alpha\end{pmatrix}_{\mathrm{NS}}\left(\begin{bmatrix}\mathrm{NN}\\\mathrm{NN}\end{bmatrix}-\begin{bmatrix}\mathrm{RR}\\\mathrm{RR}\end{bmatrix}\right)(\gamma-\beta,\gamma-\alpha;\gamma;z-A),\\
&\left(\begin{bmatrix}\mathrm{RN}\\\mathrm{RN}\end{bmatrix}-\begin{bmatrix}\mathrm{NR}\\\mathrm{NR}\end{bmatrix}\right)(\alpha,\beta;\gamma;z)\\
&\quad= \begin{pmatrix}\alpha\\\gamma-\beta\end{pmatrix}_{\mathrm{R}}\begin{pmatrix}z\\z-A\end{pmatrix}_{\mathrm{R}}\begin{pmatrix}\beta\\\gamma-\alpha\end{pmatrix}_{\mathrm{NS}}\left(\begin{bmatrix}\mathrm{RN}\\\mathrm{RN}\end{bmatrix}-\begin{bmatrix}\mathrm{NR}\\\mathrm{NR}\end{bmatrix}\right)(\gamma-\beta,\gamma-\alpha;\gamma;z-A),\\
&\left(\begin{bmatrix}\mathrm{RN}\\\mathrm{RN}\end{bmatrix}+\begin{bmatrix}\mathrm{NR}\\\mathrm{NR}\end{bmatrix}\right)(\alpha,\beta;\gamma;z)\\
&\quad= \begin{pmatrix}\alpha\\\gamma-\beta\end{pmatrix}_{\mathrm{R}}\begin{pmatrix}z\\z-A\end{pmatrix}_{\mathrm{NS}}\begin{pmatrix}\beta\\\gamma-\alpha\end{pmatrix}_{\mathrm{NS}}\left(\begin{bmatrix}\mathrm{RN}\\\mathrm{RN}\end{bmatrix}+\begin{bmatrix}\mathrm{NR}\\\mathrm{NR}\end{bmatrix}\right)(\gamma-\beta,\gamma-\alpha;\gamma;z-A),
\end{aligned} \tag{5.7}$$

where

$$A=\gamma-\alpha-\beta.$$

Reflection properties of the G functions (A.5) allow to write

$$\frac{G_{\mathrm{NS}}(z)}{G_{\mathrm{NS}}(z-A)}=\mathrm{e}^{-\frac{i\pi}{2}A(Q+A-2z)}\frac{G_{\mathrm{NS}}(Q+A-z)}{G_{\mathrm{NS}}(Q-z)},$$

and similarly for the other combinations of R/NS. Using this type of relations, formulae (5.5) and taking Fourier transform of equations (5.7) one obtains a set of integral identities analogous to the Saalschütz summation formula. In particular one gets:

$$\begin{aligned}\frac{1}{i}\int_{-i\infty}^{i\infty} d\tau\ \mathrm{e}^{i\pi\tau Q}\Bigg[&\frac{G_{\mathrm{NS}}(\tau+\alpha)G_{\mathrm{NS}}(\tau+\beta)G_{\mathrm{NS}}(\tau+\gamma)}{G_{\mathrm{NS}}(\tau+\delta)G_{\mathrm{NS}}(\tau+\alpha+\beta+\gamma-\delta+Q)G_{\mathrm{NS}}(\tau+Q)}\\ &+\frac{G_{\mathrm{R}}(\tau+\alpha)G_{\mathrm{R}}(\tau+\beta)G_{\mathrm{R}}(\tau+\gamma)}{G_{\mathrm{R}}(\tau+\delta)G_{\mathrm{R}}(\tau+\alpha+\beta+\gamma-\delta+Q)G_{\mathrm{R}}(\tau+Q)}\Bigg]\\ =2\zeta_0^{-3}\,\mathrm{e}^{\frac{i\pi}{2}\delta(Q-\delta)}&\frac{G_{\mathrm{NS}}(\alpha)G_{\mathrm{NS}}(\beta)G_{\mathrm{NS}}(\gamma)G_{\mathrm{NS}}(Q+\alpha-\delta)G_{\mathrm{NS}}(Q+\beta-\delta)G_{\mathrm{NS}}(Q+\gamma-\delta)}{G_{\mathrm{NS}}(Q+\alpha+\beta-\delta)G_{\mathrm{NS}}(Q+\alpha+\gamma-\delta)G_{\mathrm{NS}}(Q+\beta+\gamma-\delta)},\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}\frac{1}{i}\int_{-i\infty}^{i\infty} d\tau\ \mathrm{e}^{i\pi\tau Q}\Bigg[&\frac{G_{\mathrm{NS}}(\tau+\alpha)G_{\mathrm{NS}}(\tau+\beta)G_{\mathrm{R}}(\tau+\gamma)}{G_{\mathrm{R}}(\tau+\delta)G_{\mathrm{NS}}(\tau+\alpha+\beta+\gamma-\delta+Q)G_{\mathrm{NS}}(\tau+Q)}\\ &+\frac{G_{\mathrm{R}}(\tau+\alpha)G_{\mathrm{R}}(\tau+\beta)G_{\mathrm{NS}}(\tau+\gamma)}{G_{\mathrm{NS}}(\tau+\delta)G_{\mathrm{R}}(\tau+\alpha+\beta+\gamma-\delta+Q)G_{\mathrm{R}}(\tau+Q)}\Bigg]\\ =2i\zeta_0^{-3}\,\mathrm{e}^{\frac{i\pi}{2}\delta(Q-\delta)}&\frac{G_{\mathrm{NS}}(\alpha)G_{\mathrm{NS}}(\beta)G_{\mathrm{R}}(\gamma)G_{\mathrm{R}}(Q+\alpha-\delta)G_{\mathrm{R}}(Q+\beta-\delta)G_{\mathrm{NS}}(Q+\gamma-\delta)}{G_{\mathrm{R}}(Q+\alpha+\beta-\delta)G_{\mathrm{NS}}(Q+\alpha+\gamma-\delta)G_{\mathrm{NS}}(Q+\beta+\gamma-\delta)}.\end{aligned} \tag{5.9}$$

Taking in these equations the limit $\gamma \to i\infty$ we finally obtain the formulae:

$$\begin{aligned}\frac{1}{i}\int_{-i\infty}^{i\infty} d\tau\ \mathrm{e}^{i\pi\tau Q}&\left[\frac{G_{\mathrm{NS}}(\tau+\alpha)G_{\mathrm{NS}}(\tau+\beta)}{G_{\mathrm{NS}}(\tau+\delta)G_{\mathrm{NS}}(\tau+Q)}+\frac{G_{\mathrm{R}}(\tau+\alpha)G_{\mathrm{R}}(\tau+\beta)}{G_{\mathrm{R}}(\tau+\delta)G_{\mathrm{R}}(\tau+Q)}\right]\\ &=2\zeta_0^{-3}\,\mathrm{e}^{\frac{i\pi}{2}\delta(Q-\delta)}\frac{G_{\mathrm{NS}}(\alpha)G_{\mathrm{NS}}(\beta)G_{\mathrm{NS}}(Q+\alpha-\delta)G_{\mathrm{NS}}(Q+\beta-\delta)}{G_{\mathrm{NS}}(Q+\alpha+\beta-\delta)}\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\frac{1}{i}\int_{-i\infty}^{i\infty} d\tau\ \mathrm{e}^{i\pi\tau Q}&\left[\frac{G_{\mathrm{NS}}(\tau+\alpha)G_{\mathrm{NS}}(\tau+\beta)}{G_{\mathrm{R}}(\tau+\delta)G_{\mathrm{NS}}(\tau+Q)}+\frac{G_{\mathrm{R}}(\tau+\alpha)G_{\mathrm{R}}(\tau+\beta)}{G_{\mathrm{NS}}(\tau+\delta)G_{\mathrm{R}}(\tau+Q)}\right]\\ &=2i\zeta_0^{-3}\,\mathrm{e}^{\frac{i\pi}{2}\delta(Q-\delta)}\frac{G_{\mathrm{NS}}(\alpha)G_{\mathrm{NS}}(\beta)G_{\mathrm{R}}(Q+\alpha-\delta)G_{\mathrm{R}}(Q+\beta-\delta)}{G_{\mathrm{R}}(Q+\alpha+\beta-\delta)},\end{aligned} \tag{5.11}$$

which will be the main tool in the proof of the orthogonality relation presented in the next subsection.

5.2 Orthogonality and completeness relations

Define for $\xi \in i\mathbb{R}_+$

$$\begin{aligned}\langle\tau\,|^{\mathrm{N}}_{\mathrm{N}}|\,\xi\rangle &= \frac{1}{S_{\mathrm{NS}}(Q+\tau+\xi-0^+)S_{\mathrm{NS}}(Q+\tau-\xi-0^+)} &= \frac{S_{\mathrm{NS}}(\xi-\tau)}{S_{\mathrm{NS}}(Q+\tau+\xi-0^+)},\\ \langle\tau\,|^{\mathrm{R}}_{\mathrm{N}}|\,\xi\rangle &= \frac{1}{S_{\mathrm{NS}}(Q+\tau+\xi-0^+)S_{\mathrm{R}}(Q+\tau-\xi)} &= \frac{S_{\mathrm{R}}(\xi-\tau)}{S_{\mathrm{NS}}(Q+\xi+\tau-0^+)},\end{aligned}$$

etc. and

$$\langle\tau\,|^{\mathrm{N}}_{\mathrm{N}}|\,\xi\rangle^{(\epsilon)} = \frac{1}{S_{\mathrm{NS}}(Q+\tau+\xi-\epsilon)S_{\mathrm{NS}}(Q+\tau-\xi-\epsilon)} = \frac{S_{\mathrm{NS}}(\xi-\tau+\epsilon)}{S_{\mathrm{NS}}(Q+\tau+\xi-\epsilon)}.$$

Orthogonality

Using the relation between S and G functions as well as the formulae (5.10) we get:

$$\int_{-i\infty}^{i\infty} \frac{d\tau}{i} \; \Big[\langle \xi_1 \,|^{\rm N}_{\rm N}|\, \tau \rangle^{(\epsilon)} \, \langle \tau \,|^{\rm N}_{\rm N}|\, \xi_2 \rangle^{(\epsilon)} + \langle \xi_1 \,|^{\rm R}_{\rm R}|\, \tau \rangle^{(\epsilon)} \, \langle \tau \,|^{\rm R}_{\rm R}|\, \xi_2 \rangle^{(\epsilon)} \Big]$$

$$\begin{aligned} &= {\rm e}^{\frac{i\pi}{2}(\xi_2^2-\xi_1^2)} \int_{-i\infty}^{i\infty} \frac{d\tau}{i} \; {\rm e}^{i\pi Q\tau} \Bigg[\frac{G_{\rm NS}(\tau+\xi_1+\epsilon) G_{\rm NS}(\tau-\xi_1+\epsilon)}{G_{\rm NS}(Q+\tau+\xi_2-\epsilon) G_{\rm NS}(Q+\tau-\xi_2-\epsilon)} \\ &\qquad\qquad + \frac{G_{\rm R}(\tau+\xi_1+\epsilon) G_{\rm R}(\tau-\xi_1+\epsilon)}{G_{\rm R}(Q+\tau+\xi_2-\epsilon) G_{\rm R}(Q+\tau-\xi_2-\epsilon)} \Bigg] \\ &= 2\zeta_0^{-3} {\rm e}^{\frac{i\pi}{2}(\xi_2^2-\xi_1^2)-2i\pi\xi_2^2} \frac{G_{\rm NS}(2\epsilon+\xi_+) G_{\rm NS}(2\epsilon-\xi_+) G_{\rm NS}(2\epsilon+\xi_-) G_{\rm NS}(2\epsilon-\xi_-)}{G_{\rm NS}(4\epsilon)}, \end{aligned}$$

where $\xi_\pm = \xi_2 \pm \xi_1$. In view of (A.6) the r.h.s. vanishes in the limit $\epsilon \to 0$ unless $\xi_- = 0$ (we cannot have $\xi_+ = 0$ since $\Im\, \xi_i > 0, \; i = 1, 2$). Consequently:

$$\begin{aligned} &\int_{-i\infty}^{i\infty} \frac{d\tau}{i} \; \big[\langle \xi_1 \,|^{\rm N}_{\rm N}|\, \tau \rangle \, \langle \tau \,|^{\rm N}_{\rm N}|\, \xi_2 \rangle + \langle \xi_1 \,|^{\rm R}_{\rm R}|\, \tau \rangle \, \langle \tau \,|^{\rm R}_{\rm R}|\, \xi_2 \rangle \big] \\ &\qquad = 4\zeta_0^{-2} {\rm e}^{\frac{i\pi}{2}(\xi_2^2-\xi_1^2)-2i\pi\xi_2^2} \; G_{\rm NS}(\xi_+) G_{\rm NS}(-\xi_+) \; \lim_{\epsilon\to 0} \frac{2\epsilon}{\pi\left(4\epsilon^2-\xi_-^2\right)} \\ &\qquad = 4 \frac{S_{\rm NS}(2\xi_2)}{S_{\rm NS}(2\xi_2+Q)} \; \delta(p_2-p_1) = \frac{4}{\nu(\xi_2)} \; \delta(p_2-p_1), \end{aligned} \tag{5.12}$$

where $\xi_i = ip_i, \; i = 1, 2$ and

$$\nu(\xi) = -4 \sin \pi b \xi \sin \pi b^{-1} \xi.$$

Similarly:

$$\int_{-i\infty}^{i\infty} \frac{d\tau}{i} \; \Big[\langle \xi_1 \,|^{\rm N}_{\rm N}|\, \tau \rangle^{(\epsilon)} \, \langle \tau \,|^{\rm R}_{\rm R}|\, \xi_2 \rangle^{(\epsilon)} - \langle \xi_1 \,|^{\rm R}_{\rm R}|\, \tau \rangle^{(\epsilon)} \, \langle \tau \,|^{\rm N}_{\rm N}|\, \xi_2 \rangle^{(\epsilon)} \Big]$$

$$\begin{aligned} &= -i{\rm e}^{\frac{i\pi}{2}(\xi_2^2-\xi_1^2)} \int_{-i\infty}^{i\infty} \frac{d\tau}{i} \; {\rm e}^{i\pi Q\tau} \Bigg[\frac{G_{\rm NS}(\tau+\xi_1+\epsilon) G_{\rm NS}(\tau-\xi_1+\epsilon)}{G_{\rm R}(Q+\tau+\xi_2-\epsilon) G_{\rm R}(Q+\tau-\xi_2-\epsilon)} \\ &\qquad\qquad + \frac{G_{\rm R}(\tau+\xi_1+\epsilon) G_{\rm R}(\tau-\xi_1+\epsilon)}{G_{\rm NS}(Q+\tau+\xi_2-\epsilon) G_{\rm NS}(Q+\tau-\xi_2-\epsilon)} \Bigg] \\ &= -2i\zeta_0^{-3} {\rm e}^{\frac{i\pi}{2}(\xi_2^2-\xi_1^2)-2i\pi\xi_2^2} \frac{G_{\rm R}(2\epsilon+\xi_+) G_{\rm R}(2\epsilon-\xi_+) G_{\rm R}(2\epsilon+\xi_-) G_{\rm R}(2\epsilon-\xi_-)}{G_{\rm NS}(4\epsilon)}. \end{aligned}$$

The function $G_{\rm R}(x)$ is regular in the vicinity of the imaginary axis and taking the limit $\epsilon \to 0$ we have for $\xi_1, \xi_2 \in i\mathbb{R}_+$:

$$\int_{-i\infty}^{i\infty} \frac{d\tau}{i} \; \big[\langle \xi_1 \,|^{\rm N}_{\rm N}|\, \tau \rangle \, \langle \tau \,|^{\rm R}_{\rm R}|\, \xi_2 \rangle - \langle \xi_1 \,|^{\rm R}_{\rm R}|\, \tau \rangle \, \langle \tau \,|^{\rm N}_{\rm N}|\, \xi_2 \rangle \big] = 0. \tag{5.13}$$

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Completeness

Define

$$\nu_\epsilon(\xi) \;=\; -4\sin\left(\pi b_\epsilon \xi\right)\sin\left(\pi b_\epsilon^{-1}\xi\right), \qquad b_\epsilon^{\pm 1} \;=\; b^{\pm 1}-\epsilon, \qquad \lambda_\epsilon \;=\; \lambda+\epsilon,$$

and consider an integral:

$$\begin{aligned}
\mathcal{I}^\epsilon(\lambda,\rho) &= \int_{-i\infty}^{i\infty}\frac{d\tau}{i}\int_{-i\infty}^{i\infty}\frac{d\xi}{i}\;\nu_\epsilon(\xi)\left[\langle \tau-\lambda_\epsilon\,|^{\rm N}_{\rm N}|\,\xi\rangle\,\langle \xi\,|^{\rm N}_{\rm N}|\,\tau\rangle + \langle \tau-\lambda_\epsilon\,|^{\rm R}_{\rm R}|\,\xi\rangle\,\langle \xi\,|^{\rm R}_{\rm R}|\,\tau\rangle\right]{\rm e}^{-i\pi\rho\tau}\\
&= \sum_{k=1}^{4}\frac{(-1)^k}{2}\left[\int_{-i\infty}^{i\infty}\frac{du}{i}\;{\rm e}^{-\frac{i\pi u}{2}(\rho-\rho_k)}\frac{S_{\rm NS}(u)}{S_{\rm NS}(u-\lambda_\epsilon+Q)}\right]\left[\int_{-i\infty}^{i\infty}\frac{dv}{i}\;{\rm e}^{\frac{i\pi v}{2}(\rho+\rho_k)}\frac{S_{\rm NS}(v+\lambda_\epsilon)}{S_{\rm NS}(v+Q)}\right]\\
&\quad+\sum_{k=1}^{4}\frac{(-1)^k}{2}\left[\int_{-i\infty}^{i\infty}\frac{du}{i}\;{\rm e}^{-\frac{i\pi u}{2}(\rho-\rho_k)}\frac{S_{\rm R}(u)}{S_{\rm R}(u-\lambda_\epsilon+Q)}\right]\left[\int_{-i\infty}^{i\infty}\frac{dv}{i}\;{\rm e}^{\frac{i\pi v}{2}(\rho+\rho_k)}\frac{S_{\rm R}(v+\lambda_\epsilon)}{S_{\rm R}(v+Q)}\right],
\end{aligned}$$

where $u=\tau+\xi,\; v=\tau-\xi$, and $\rho_1=-\rho_3=b+b^{-1}-2\epsilon,\;\rho_2=-\rho_4=b-b^{-1}$. It can be calculated by means of the formulae presented in the appendix A and we get:

$$\mathcal{I}^\epsilon(\lambda,\rho) \;=\; \mathcal{I}^\epsilon_1(\lambda,\rho)-\mathcal{I}^\epsilon_2(\lambda,\rho)+\mathcal{I}^\epsilon_3(\lambda,\rho)-\mathcal{I}^\epsilon_4(\lambda,\rho)$$

where

$$\begin{aligned}
\mathcal{I}^\epsilon_1(\lambda,\rho) &= 2S^2_{\rm NS}(\lambda_\epsilon)\frac{G_{\rm NS}\left(\frac{\rho-\lambda_\epsilon}{2}+\epsilon\right)G_{\rm NS}\left(Q+\frac{\rho-\lambda_\epsilon}{2}-\epsilon\right)}{G_{\rm NS}\left(\frac{\rho+\lambda_\epsilon}{2}+\epsilon\right)G_{\rm NS}\left(Q+\frac{\rho+\lambda_\epsilon}{2}-\epsilon\right)},\\
\mathcal{I}^\epsilon_2(\lambda,\rho) &= 2S^2_{\rm NS}(\lambda_\epsilon)\frac{G_{\rm R}\left(b+\frac{\rho-\lambda_\epsilon}{2}\right)G_{\rm R}\left(b^{-1}+\frac{\rho-\lambda_\epsilon}{2}\right)}{G_{\rm R}\left(b+\frac{\rho+\lambda_\epsilon}{2}\right)G_{\rm R}\left(b^{-1}+\frac{\rho+\lambda_\epsilon}{2}\right)},\\
\mathcal{I}^\epsilon_3(\lambda,\rho) &= 2S^2_{\rm NS}(\lambda_\epsilon)\frac{G_{\rm R}\left(\frac{\rho-\lambda_\epsilon}{2}+\epsilon\right)G_{\rm R}\left(Q+\frac{\rho-\lambda_\epsilon}{2}-\epsilon\right)}{G_{\rm R}\left(\frac{\rho+\lambda_\epsilon}{2}+\epsilon\right)G_{\rm R}\left(Q+\frac{\rho+\lambda_\epsilon}{2}-\epsilon\right)},\\
\mathcal{I}^\epsilon_4(\lambda,\rho) &= 2S^2_{\rm NS}(\lambda_\epsilon)\frac{G_{\rm NS}\left(b+\frac{\rho-\lambda_\epsilon}{2}\right)G_{\rm NS}\left(b^{-1}+\frac{\rho-\lambda_\epsilon}{2}\right)}{G_{\rm NS}\left(b+\frac{\rho+\lambda_\epsilon}{2}\right)G_{\rm NS}\left(b^{-1}+\frac{\rho+\lambda_\epsilon}{2}\right)}.
\end{aligned}$$

It is immediate to check with the help of relations (A.4) that outside of the singularities of the functions involved $\lim\limits_{\epsilon\to 0}\mathcal{I}^\epsilon(\lambda,\rho)=0$. However, since for $\epsilon\to 0$ some of the singularities approach the lines $\Im\rho=0$ and $\Im\lambda=0$ (the integration contours for the distribution $\mathcal{I}(\lambda,\rho)$), we have to be more careful. The correct way of proceeding is analogous to the calculation in section 4.3.

For $\varphi(\lambda,\rho)$ being a test function consider

$$\varphi_i \;=\; \lim_{\epsilon\to 0}\int_{-i\infty}^{i\infty}\frac{d\lambda}{i}\int_{-i\infty}^{i\infty}\frac{d\rho}{i}\;\mathcal{I}^\epsilon_i(\lambda,\rho)\varphi(\lambda,\rho). \tag{5.14}$$

$\mathcal{I}_i(\lambda,\rho)$ have poles at imaginary λ and ρ axis. From the form of $\mathcal{I}_2^\epsilon$, $\mathcal{I}_3^\epsilon$ and $\mathcal{I}_4^\epsilon$ it is clear that all one has to do to define the limit $\epsilon \to 0$ of these terms is to deform the contour of integration over λ such that it avoids the singularity at $\lambda = -\epsilon$. Since no poles pinching the integration contours appear, there are no contributions from the residues and

$$\varphi_i \;=\; \int\limits_{\mathcal{C}_\lambda}\frac{d\lambda}{i}\int\limits_{\mathcal{C}_\rho}\frac{d\rho}{i}\;\mathcal{I}_i(\lambda,\rho)\varphi(\lambda,\rho), \qquad i=2,3,4,$$

where $\mathcal{C}_\lambda$ and $\mathcal{C}_\rho$ denote the deformed contours.

The situation for $\mathcal{I}_1^\epsilon$ is different. In the complex ρ plane a function $G_{\rm NS}\left(\frac{\rho-\lambda_\epsilon}{2}+\epsilon\right)$ has a pole at $\rho=\rho_- = \lambda_\epsilon - 2\epsilon = \lambda-\epsilon$ (to the left of the integration contour $\rho\in i\mathbb{R}$), while a function $G_{\rm NS}\left(Q+\frac{\rho+\lambda_\epsilon}{2}-\epsilon\right)^{-1}$ has a pole at $\rho=\rho_+ = -\lambda_\epsilon+2\epsilon = -\lambda+\epsilon$ (to the right of the integration contour). For $\lambda,\epsilon\to 0$ these poles collide. Choosing to deform the contour past the pole at $\rho=\rho_-$, taking into account (A.6) and the relation

$$\begin{aligned}
&\zeta_0 \lim_{\epsilon\to 0}\int\limits_{-i\infty}^{i\infty}\frac{d\lambda}{i}\;\frac{S_{\rm NS}^2(\lambda_\epsilon)G_{\rm NS}(Q-2\epsilon)}{G_{\rm NS}(\lambda_\epsilon)G_{\rm NS}(Q+\lambda_\epsilon-2\epsilon)}\varphi(\lambda,\lambda_\epsilon-2\epsilon)\\
&\qquad = \lim_{\epsilon\to 0}\int\limits_{-i\infty}^{i\infty}\frac{d\lambda}{i}\;\mathrm{e}^{\frac{i\pi}{2}Q\lambda}\;\frac{S_{\rm NS}(\epsilon+\lambda)S_{\rm NS}(\epsilon-\lambda)}{S_{\rm NS}(2\epsilon)}\varphi(\lambda,\lambda-\epsilon)\\
&\qquad = \lim_{\epsilon\to 0}\int\limits_{-i\infty}^{i\infty}\frac{d\lambda}{i}\;\mathrm{e}^{\frac{i\pi}{2}Q\lambda}\;\frac{2\epsilon}{\pi\left(\epsilon^2-\lambda^2\right)}\varphi(\lambda,\lambda-\epsilon)\;=\;2\varphi(0,0),
\end{aligned}$$

we get

$$\varphi_1 = \int\limits_{\mathcal{C}_\lambda}\frac{d\lambda}{i}\int\limits_{\mathcal{C}_\rho}\frac{d\rho}{i}\;\mathcal{I}_1(\lambda,\rho)\varphi(\lambda,\rho)+16\varphi(0,0),$$

and finally

$$\begin{aligned}
\sum_{k=1}^{4}\varphi_k &= 16\varphi(0,0)+\int\limits_{\mathcal{C}_\lambda}\frac{d\lambda}{i}\int\limits_{\mathcal{C}_\rho}\frac{d\rho}{i}\;\sum_{k=1}^{k}(-1)^{k-1}\mathcal{I}_k(\lambda,\rho)\varphi(\lambda,\rho)\\
&= 16\varphi(0,0)+\int\limits_{\mathcal{C}_\lambda}\frac{d\lambda}{i}\int\limits_{\mathcal{C}_\rho}\frac{d\rho}{i}\;\left[\sum_{k=1}^{k}(-1)^{k-1}\mathcal{I}_k(\lambda,\rho)\right]\varphi(\lambda,\rho)\;=\;16\varphi(0,0),
\end{aligned}$$

what proves the equality

$$\lim_{\epsilon\to 0}\mathcal{I}_1^\epsilon(\lambda,\rho)\;=\;16\delta(\lambda)\delta(\rho).$$

Taking the inverse Fourier transform we have

$$\int\limits_{-i\infty}^{i\infty}\frac{d\xi}{i}\;\nu(\xi)\Big(\left\langle\eta-\lambda\left|{}^{\rm N}_{\rm N}\right|\xi\right\rangle\left\langle\xi\left|{}^{\rm N}_{\rm N}\right|\eta\right\rangle+\left\langle\eta-\lambda\left|{}^{\rm R}_{\rm R}\right|\xi\right\rangle\left\langle\xi\left|{}^{\rm R}_{\rm R}\right|\eta\right\rangle\Big)=\int\limits_{-i\infty}^{i\infty}\frac{d\rho}{2i}\;I_1(\lambda,\rho)\,\mathrm{e}^{i\pi\rho\eta}=8\delta(\lambda). \tag{5.15}$$

Analogous computation gives:

$$\frac{1}{2S_{\rm R}^2(\lambda_\epsilon)}\int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\int\limits_{-i\infty}^{i\infty}\frac{d\xi}{i}\,\nu_\epsilon(\xi)\left[\langle\tau-\lambda_\epsilon\,|^{\rm R}_{\rm R}|\,\xi\rangle\,\langle\xi\,|^{\rm N}_{\rm N}|\,\tau\rangle-\langle\tau-\lambda_\epsilon\,|^{\rm N}_{\rm N}|\,\xi\rangle\,\langle\xi\,|^{\rm R}_{\rm R}|\,\tau\rangle\right]{\rm e}^{-i\pi\rho\tau}$$
$$=\frac{G_{\rm NS}\left(\frac{\rho-\lambda_\epsilon}{2}+\epsilon\right)}{G_{\rm R}\left(\frac{\rho+\lambda_\epsilon}{2}+\epsilon\right)}\frac{G_{\rm NS}\left(Q+\frac{\rho-\lambda_\epsilon}{2}-\epsilon\right)}{G_{\rm R}\left(Q+\frac{\rho+\lambda_\epsilon}{2}-\epsilon\right)}-\frac{G_{\rm R}\left(b+\frac{\rho-\lambda_\epsilon}{2}\right)}{G_{\rm NS}\left(b+\frac{\rho+\lambda_\epsilon}{2}\right)}\frac{G_{\rm R}\left(b^{-1}+\frac{\rho-\lambda_\epsilon}{2}\right)}{G_{\rm NS}\left(b^{-1}+\frac{\rho+\lambda_\epsilon}{2}\right)}$$
$$+\frac{G_{\rm R}\left(\frac{\rho-\lambda_\epsilon}{2}+\epsilon\right)}{G_{\rm NS}\left(\frac{\rho+\lambda_\epsilon}{2}+\epsilon\right)}\frac{G_{\rm R}\left(Q+\frac{\rho-\lambda_\epsilon}{2}-\epsilon\right)}{G_{\rm NS}\left(Q+\frac{\rho+\lambda_\epsilon}{2}-\epsilon\right)}-\frac{G_{\rm NS}\left(b+\frac{\rho-\lambda_\epsilon}{2}\right)}{G_{\rm R}\left(b+\frac{\rho+\lambda_\epsilon}{2}\right)}\frac{G_{\rm NS}\left(b^{-1}+\frac{\rho-\lambda_\epsilon}{2}\right)}{G_{\rm R}\left(b^{-1}+\frac{\rho+\lambda_\epsilon}{2}\right)}.$$

Since in this case there are no poles pinching the integration contours (and $S_{\rm R}(\lambda)$ is regular for $\lambda\in i\mathbb{R}$) we get:

$$\int\limits_{-i\infty}^{i\infty}\frac{d\xi}{i}\,\nu(\xi)\Big(\langle\eta-\lambda\,|^{\rm R}_{\rm R}|\,\xi\rangle\,\langle\xi\,|^{\rm N}_{\rm N}|\,\eta\rangle-\langle\eta-\lambda\,|^{\rm N}_{\rm N}|\,\xi\rangle\,\langle\xi\,|^{\rm R}_{\rm R}|\,\eta\rangle\Big)\;=\;0. \tag{5.16}$$

6. Discussion

Construction of the fusion matrix presented in this paper can be placed on a more firm ground by establishing its relation (in the spirit of [8, 9]) to the representation theory of quantum groups. A natural candidate (see [24]) is $U_q({\rm osp}(2|1))$: q-deformed universal enveloping algebra of osp(2|1) with a deformation parameter $q={\rm e}^{i\pi b^2}$ [25]. Indeed, generalizing the construction of [9] one can define on $V=L^2(\mathbb{R})\times L^2(\mathbb{R})$ a continuous series of representations of $U_q({\rm osp}(2|1))$ with the generators given by:

$$v_\alpha^{(+)}={\rm e}^{\pi bx}\begin{pmatrix}0 & [\delta_x+Q-\alpha]_{\rm R}\\ [\delta_x+Q-\alpha]_{\rm NS} & 0\end{pmatrix},\quad v_\alpha^{(-)}={\rm e}^{-\pi bx}\begin{pmatrix}0 & [\delta_x+\alpha-Q]_{\rm R}\\ [\delta_x+\alpha-Q]_{\rm NS} & 0\end{pmatrix},$$

and

$$K_\alpha={\sf T}_x^{\frac{ib}{2}}\sigma_0,$$

where

$${\sf T}_x^a\,f(x)\;=\;f\,(x+a)$$

and

$$[\delta_x+a]_{\rm R}=\frac{{\rm e}^{\frac{i\pi ba}{2}}{\sf T}_x^{\frac{ib}{2}}-{\rm e}^{-\frac{i\pi ba}{2}}{\sf T}_x^{-\frac{ib}{2}}}{{\rm e}^{\frac{i\pi b^2}{2}}-{\rm e}^{-\frac{i\pi b^2}{2}}},\qquad [\delta_x+a]_{\rm NS}=\frac{{\rm e}^{\frac{i\pi ba}{2}}{\sf T}_x^{\frac{ib}{2}}+{\rm e}^{-\frac{i\pi ba}{2}}{\sf T}_x^{-\frac{ib}{2}}}{{\rm e}^{\frac{i\pi b^2}{2}}+{\rm e}^{-\frac{i\pi b^2}{2}}}.$$

This representation possesses many virtues analogous to those of the representation of $U_q({\rm sl}(2,\mathbb{R}))$ studied in [9], which proved to be crucial in relating $U_q({\rm sl}(2,\mathbb{R}))$ to the Liouville theory. For instance, replacing in $v_\alpha^{(\pm)}$ and K_α the parameter b with b^{-1}, we obtain a continuous family of representations (on the same space V) of generators of a "dual" quantum supergroup $U_{\tilde{q}}(\mathrm{osp}(2|1))$ with the deformation parameter $\tilde{q} = \mathrm{e}^{i\pi b^{-2}}$. Since

$$\mathsf{T}^{ib}_{\omega}\,\frac{S_{\rm NS}(\alpha-i\omega)}{S_{\rm NS}(\bar\alpha-i\omega)} = \frac{[\alpha-i\omega]_{\rm NS}}{[\bar\alpha-i\omega]_{\rm NS}}\,\frac{S_{\rm R}(\alpha-i\omega)}{S_{\rm R}(\bar\alpha-i\omega)}\,\mathsf{T}^{ib}_{\omega},$$
$$\mathsf{T}^{ib}_{\omega}\,\frac{S_{\rm R}(\alpha-i\omega)}{S_{\rm R}(\bar\alpha-i\omega)} = \frac{[\alpha-i\omega]_{\rm R}}{[\bar\alpha-i\omega]_{\rm R}}\,\frac{S_{\rm NS}(\alpha-i\omega)}{S_{\rm NS}(\bar\alpha-i\omega)}\,\mathsf{T}^{ib}_{\omega},$$

where

$$[a]_{\rm R} = \frac{\sin\frac{\pi b a}{2}}{\sin\frac{\pi b^2}{2}}, \qquad [a]_{\rm NS} = \frac{\cos\frac{\pi b a}{2}}{\cos\frac{\pi b^2}{2}},$$

it is easy to see that for a unitary matrix

$$\widetilde{\mathcal{I}}_\alpha = \begin{pmatrix} \frac{S_{\rm NS}(\alpha-i\omega)}{S_{\rm NS}(\bar\alpha-i\omega)} & 0 \\ 0 & \frac{S_{\rm R}(\alpha-i\omega)}{S_{\rm R}(\bar\alpha-i\omega)} \end{pmatrix}$$

and $\widetilde{\mathcal{O}}_\alpha = \widetilde{v}^{(\pm)}_{\alpha,}\,\widetilde{K}_\alpha$ being Fourier-transformed generators $v^{(\pm)}_{\alpha,}\,K_\alpha$, we have:

$$\widetilde{\mathcal{O}}_{Q-\alpha}\,\widetilde{\mathcal{I}}_\alpha = \widetilde{\mathcal{I}}_\alpha\,\widetilde{\mathcal{O}}_\alpha,$$

what proves equivalence of representations $\mathcal{O}_\alpha$ and $\mathcal{O}_{Q-\alpha}$. Moreover, it turns out to be possible to express a Clebsch-Gordan coefficients for this representation through a ratios of special functions $S_{\rm NS,R}$ and to relate the matrix $\mathbf{F}$ to the Racah-Wigner coefficients (the main technical tools for this construction are provided by the formulae from section 5). This results will be reported elsewhere [26].

Let us conclude with a few remarks.

Results from the quantum Liouville theory have a number of applications, to name only quantization of Teichmüller space of Riemann surfaces [27] and relation between Liouville theory and the H_3^+ WZNW model [28–32]. Extension of these results with the help of the results of the present paper seem to be both possible and interesting.

The fusion matrix of conformal blocks is related (through the "renaming" of variables) to the three point correlation function of the boundary operators in the Liouville theory [33, 34]. Once the fusion matrix of the NS blocks is known, it seems not to be difficult to generalize this link and calculate the (so far unknown) three point function of the boundary operators in the NS sector of the supersymmetric Liouville theory.

Last but not least, it is plausible that the result of the present paper will allow to better understand some general properties of the $N = 1$ super-conformal Ramond blocks [10].

Acknowledgments

L.H. would like to thank the Alexander von Humboldt Foundation for the support during his stay at Bonn, R. Flume for numerous discussions and encouragement and the faculty of the Physics Institute if the Bonn University (especially V. Rittenberg) for providing a highly stimulating scientific environment and for a warm hospitality.

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A. Definitions and main properties of the Barnes functions

For $\Re x > 0$ the Barnes double gamma function has an integral representation of the form:

$$\log\,\Gamma_b(x) \;=\; \int\limits_0^\infty \frac{dt}{t}\left[\frac{\mathrm{e}^{-xt}-\mathrm{e}^{-\frac{Q}{2}t}}{\left(1-\mathrm{e}^{-tb}\right)\left(1-\mathrm{e}^{-t/b}\right)} - \frac{\left(\frac{Q}{2}-x\right)^2}{2\mathrm{e}^t} - \frac{\frac{Q}{2}-x}{t}\right].$$

It satisfies functional relations

$$\begin{aligned} \Gamma_b(x+b) &= \frac{\sqrt{2\pi}\, b^{bx-\frac{1}{2}}}{\Gamma(bx)}\Gamma_b(x), \\ \Gamma_b\left(x+b^{-1}\right) &= \frac{\sqrt{2\pi}\, b^{-\frac{x}{b}+\frac{1}{2}}}{\Gamma(\frac{x}{b})}\Gamma_b(x), \end{aligned} \tag{A.1}$$

and can be analytically continued to the whole complex x plane as a meromorphic function with no zeroes and with poles located at $x = -mb - n\frac{1}{b},\ m,n \in \mathbb{N}$. Relations (A.1) allow to calculate residues of these poles in terms of $\Gamma_b(Q)$; for instance for $x \to 0$:

$$\Gamma_b(x) \;=\; \frac{\Gamma_b(Q)}{2\pi x} + \mathcal{O}(1).$$

It is convenient to introduce

$$\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}, \qquad S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}, \qquad G_b(x) = \mathrm{e}^{-\frac{i\pi}{2}x(Q-x)}S_b(x), \tag{A.2}$$

and, borrowing the notation from [21], to denote:

$$\begin{aligned} \Gamma_{\rm NS}(x) &= \Gamma_b\left(\frac{x}{2}\right)\Gamma_b\left(\frac{x+Q}{2}\right), \qquad & \Gamma_{\rm R}(x) &= \Gamma_b\left(\frac{x+b}{2}\right)\Gamma_b\left(\frac{x+b^{-1}}{2}\right), \\ \Upsilon_{\rm NS}(x) &= \Upsilon_b\left(\frac{x}{2}\right)\Upsilon_b\left(\frac{x+Q}{2}\right), \qquad & \Upsilon_{\rm R}(x) &= \Upsilon_b\left(\frac{x+b}{2}\right)\Upsilon_b\left(\frac{x+b^{-1}}{2}\right), \end{aligned} \tag{A.3}$$

etc.

Using relations (A.1) and definitions (A.2), (A.3) one can easily establish basic properties of these functions. They include:

Relations between S and G functions

$$\begin{aligned} G_{\rm NS}(x) &= \zeta_0\,\mathrm{e}^{-\frac{i\pi}{4}x(Q-x)}S_{\rm NS}(x), \\ G_{\rm R}(x) &= \mathrm{e}^{-\frac{i\pi}{4}}\zeta_0\,\mathrm{e}^{-\frac{i\pi}{4}x(Q-x)}S_{\rm R}(x), \end{aligned}$$

where $\zeta_0 \;=\; \mathrm{e}^{-\frac{i\pi Q^2}{8}}$.

Shift relations

$$\begin{aligned} S_{\rm NS}(x+b^{\pm 1}) &= 2\cos\left(\frac{\pi b^{\pm1}x}{2}\right)S_{\rm R}(x), \qquad & S_{\rm R}(x+b^{\pm 1}) &= 2\sin\left(\frac{\pi b^{\pm1}x}{2}\right)S_{\rm NS}(x), \\ G_{\rm NS}(x+b^{\pm 1}) &= \left(1+\mathrm{e}^{i\pi b^{\pm1}x}\right)G_{\rm R}(x), \qquad & G_{\rm R}(x+b^{\pm 1}) &= \left(1-\mathrm{e}^{i\pi b^{\pm1}x}\right)G_{\rm NS}(x). \end{aligned} \tag{A.4}$$

Reflection properties

$$S_{\rm NS}(x)S_{\rm NS}(Q-x) \;=\; S_{\rm R}(x)S_{\rm R}(Q-x)=1$$

and consequently:

$$\begin{aligned} G_{\rm NS}(x)G_{\rm NS}(Q-x) \;&=\; \zeta_0^2\,{\rm e}^{-\frac{i\pi}{2}x(Q-x)}, \\ G_{\rm R}(x)G_{\rm R}(Q-x) \;&=\; {\rm e}^{-\frac{i\pi}{2}}\zeta_0^2\,{\rm e}^{-\frac{i\pi}{2}x(Q-x)}. \end{aligned} \tag{A.5}$$

Asymptotic behavior

$$\begin{aligned} G_{\rm NS}(x) \;&\to\; \begin{cases} 1, & x\to+i\infty, \\ \zeta_0^2\,{\rm e}^{-\frac{i\pi}{2}x(Q-x)}, & x\to-i\infty, \end{cases} \\ G_{\rm R}(x) \;&\to\; \begin{cases} 1, & x\to+i\infty, \\ {\rm e}^{-\frac{i\pi}{2}}\zeta_0^2\,{\rm e}^{-\frac{i\pi}{2}x(Q-x)}, & x\to-i\infty. \end{cases} \end{aligned}$$

Zeroes and poles

$$\begin{aligned} S_{\rm NS}(x)=0 \quad&\Leftrightarrow\quad x=Q+mb+nb^{-1}, \quad m,n\in\mathbb{Z}_{\geq 0}, \quad m+n\in 2\mathbb{Z}, \\ S_{\rm R}(x)=0 \quad&\Leftrightarrow\quad x=Q+mb+nb^{-1}, \quad m,n\in\mathbb{Z}_{\geq 0}, \quad m+n\in 2\mathbb{Z}+1, \\ S_{\rm NS}(x)^{-1}=0 \quad&\Leftrightarrow\quad x=-mb-nb^{-1}, \quad m,n\in\mathbb{Z}_{\geq 0}, \quad m+n\in 2\mathbb{Z}, \\ S_{\rm R}(x)^{-1}=0 \quad&\Leftrightarrow\quad x=-mb-nb^{-1}, \quad m,n\in\mathbb{Z}_{\geq 0}, \quad m+n\in 2\mathbb{Z}+1. \end{aligned}$$

Basic residues

$$\lim_{x\to 0}\; x\,S_{\rm NS}(x) \;=\; \frac{1}{\pi}, \qquad \lim_{x\to 0}\; x\,G_{\rm NS}(x) = \frac{1}{\pi}\zeta_0. \tag{A.6}$$

B. Integral formulae for the functions $S_{\rm NS,R}(x)$.

In this appendix we have collected the integral formulae satisfied by the rations of two of the special functions $S_{\rm NS,R}(x)$. They can be derived from (5.5) and read:

$$\begin{aligned} \int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\;{\rm e}^{\frac{i\pi}{2}\tau\beta}\frac{S_{\rm R}(\tau+\alpha)}{S_{\rm NS}(\tau+Q)} \;&=\; S_{\rm R}(\alpha)\left[\frac{G_{\rm NS}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\rm R}\left(\frac{Q+\beta+\alpha}{2}\right)}+\frac{G_{\rm R}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\rm NS}\left(\frac{Q+\beta+\alpha}{2}\right)}\right], \\ \int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\;{\rm e}^{\frac{i\pi}{2}\tau\beta}\frac{S_{\rm NS}(\tau+\alpha)}{S_{\rm R}(\tau+Q)} \;&=\; -iS_{\rm R}(\alpha)\left[\frac{G_{\rm NS}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\rm R}\left(\frac{Q+\beta+\alpha}{2}\right)}-\frac{G_{\rm R}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\rm NS}\left(\frac{Q+\beta+\alpha}{2}\right)}\right], \end{aligned} \tag{B.1}$$

and

$$\begin{aligned} \int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\;{\rm e}^{\frac{i\pi}{2}\tau\beta}\frac{S_{\rm NS}(\tau+\alpha)}{S_{\rm NS}(\tau+Q)} \;&=\; S_{\rm NS}(\alpha)\left[\frac{G_{\rm NS}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\rm N}\left(\frac{Q+\beta+\alpha}{2}\right)}+\frac{G_{\rm R}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\rm R}\left(\frac{Q+\beta+\alpha}{2}\right)}\right], \\ \int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\;{\rm e}^{\frac{i\pi}{2}\tau\beta}\frac{S_{\rm R}(\tau+\alpha)}{S_{\rm R}(\tau+Q)} \;&=\; S_{\rm NS}(\alpha)\left[\frac{G_{\rm NS}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\rm NS}\left(\frac{Q+\beta+\alpha}{2}\right)}-\frac{G_{\rm R}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\rm R}\left(\frac{Q+\beta+\alpha}{2}\right)}\right]. \end{aligned} \tag{B.2}$$

We shall also need expression for the complex conjugations of the l.h.s. They are of the form:

$$\begin{aligned}
\int_{-i\infty}^{i\infty}\frac{d\tau}{i}\,\mathrm{e}^{-\frac{i\pi}{2}\tau\beta}\frac{S_{\mathrm{R}}(\tau)}{S_{\mathrm{NS}}(\tau-\alpha+Q)} &= -i\,S_{\mathrm{R}}(\alpha)\left[\frac{G_{\mathrm{NS}}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\mathrm{R}}\left(\frac{Q+\beta+\alpha}{2}\right)}-\frac{G_{\mathrm{R}}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\mathrm{NS}}\left(\frac{Q+\beta+\alpha}{2}\right)}\right],\\
\int_{-i\infty}^{i\infty}\frac{d\tau}{i}\,\mathrm{e}^{-\frac{i\pi}{2}\tau\beta}\frac{S_{\mathrm{NS}}(\tau)}{S_{\mathrm{R}}(\tau-\alpha+Q)} &= S_{\mathrm{R}}(\alpha)\left[\frac{G_{\mathrm{NS}}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\mathrm{R}}\left(\frac{Q+\beta+\alpha}{2}\right)}+\frac{G_{\mathrm{R}}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\mathrm{NS}}\left(\frac{Q+\beta+\alpha}{2}\right)}\right],
\end{aligned}\tag{B.3}$$

and

$$\begin{aligned}
\int_{-i\infty}^{i\infty}\frac{d\tau}{i}\,\mathrm{e}^{-\frac{i\pi}{2}\tau\beta}\frac{S_{\mathrm{NS}}(\tau)}{S_{\mathrm{NS}}(\tau-\alpha+Q)} &= S_{\mathrm{NS}}(\alpha)\left[\frac{G_{\mathrm{NS}}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\mathrm{NS}}\left(\frac{Q+\beta+\alpha}{2}\right)}+\frac{G_{\mathrm{R}}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\mathrm{R}}\left(\frac{Q+\beta+\alpha}{2}\right)}\right],\\
\int_{-i\infty}^{i\infty}\frac{d\tau}{i}\,\mathrm{e}^{-\frac{i\pi}{2}\tau\beta}\frac{S_{\mathrm{R}}(\tau)}{S_{\mathrm{R}}(\tau-\alpha+Q)} &= S_{\mathrm{NS}}(\alpha)\left[\frac{G_{\mathrm{NS}}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\mathrm{NS}}\left(\frac{Q+\beta+\alpha}{2}\right)}-\frac{G_{\mathrm{R}}\left(\frac{Q+\beta-\alpha}{2}\right)}{G_{\mathrm{R}}\left(\frac{Q+\beta+\alpha}{2}\right)}\right].
\end{aligned}\tag{B.4}$$

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